

As we have done in other situations, let's try to characterize G in a way which is indep. of the eigenfn expansion.

1) For $t > 0$ G satisfies wave eqn $(\frac{1}{c^2} \partial_t^2 - \nabla_x^2) G(\vec{x}, \vec{y}, t) = 0$

2) BC $G(\vec{x}, \vec{y}, t) \Big|_{\vec{x} \in \mathcal{C}} = 0$

3) IC's: $G(\vec{x}, \vec{y}, t=0) = 0$

$$\partial_t G(\vec{x}, \vec{y}, t) = \sum_n \frac{\cos(\omega_n t)}{N_n^2} \phi_n(x) \phi_n(y)$$

$$\xrightarrow{t \rightarrow 0^+} \sum_n \frac{\phi_n(x) \phi_n(y)}{N_n^2} = \delta^{(2)}(\vec{x} - \vec{y}) \text{ by completeness}$$

Then is the same IVP we found for 1-dim $G(x, y, t)$ on $\mathbb{R}$ in last class. In all situations, the Green's fn for the wave eqn is the unique sol of the PDE which describes the response of the medium to a $\delta(\vec{x} - \vec{y})$ spike in initial velocity data.

Green's fn

We also see that the solution of the general IVP has same structure for both 1-dim ∞ stretched string and the finite drumhead. In fact this structure is universal. It holds for ∞ or finite region (Derrick) problems in any dimension.

The special case for $G(\vec{x}, \vec{y}, t)$ is also universal.

Compare our treatment with Ch 11 of text. Haberman uses slightly different def of G , and we will use the term causal Grn. fn for this:

$$G_c(\vec{x}, \vec{y}, t) = \theta(t) G(\vec{x}, \vec{y}, t) = \begin{cases} G(\vec{x}, \vec{y}, t) & t > 0 \\ 0 & t < 0 \end{cases}$$

a simple extension of prev. G into the past.

Ex: Use $\partial_t \theta(t) = \delta(t)$ and prev. info on G to show that G_c satisfies $[\frac{1}{c^2} \partial_t^2 - \nabla_x^2] G_c(\vec{x}, \vec{y}, t) = \frac{1}{c^2} \delta(t) \delta(\vec{x} - \vec{y})$

So G_c is a response of the medium to an external pt. source at $\vec{x} = \vec{y}$ which is instantaneous in time.

We can use G_c to write a sol of inhom. wave eqn with general source.

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right] u = \frac{1}{c^2} g(\vec{x}, t)$$

$$\begin{aligned} u(\vec{x}, t) &= \int_{\mathcal{R}} d^2 \vec{y} \int_0^\infty dt' G_c(\vec{x}, \vec{y}, t-t') g(\vec{y}, t') \\ &= \int_{\mathcal{R}} d^2 \vec{y} \int_0^t dt' G(\vec{x}, \vec{y}, t-t') g(\vec{y}, t') \end{aligned}$$

We have introduced source time t' (as we did for half flow with source).
Change of upper limit in 2nd form comes from $\Theta(t-t')$ factor in G_c .
Our sol satisfies causality — only values of source time $t' \leq t$ influence the response at time t .

Since spectral form of G is universal, we can easily generalize from discrete spectrum of ∇^2 on finite $\mathcal{R}$ to the contin. spect. of plane waves for full $\mathbb{R}^d$.

Dictionary for translation:

$$\frac{1}{N_n} \phi_n(\vec{x}) \rightarrow \phi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^{d/2}} e^{i\vec{k} \cdot \vec{x}}$$

$$\frac{1}{N_n} \phi_n(\vec{y}) \rightarrow \phi_{\vec{k}}^*(\vec{y}) = \frac{1}{(2\pi)^{d/2}} e^{-i\vec{k} \cdot \vec{y}}$$

$$\sum_n \rightarrow \int d^d k$$

$$\omega_n = \sqrt{\lambda_n} \rightarrow |k|$$

So the Green's fn for the wave eq on full $\mathbb{R}^d$ is:

$$G^{(d)}(\vec{x} - \vec{y}, t) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{c|k|} \sin(1k/ct) e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

Unless you are a superstring theorist, the most interesting cases are $d=1, 2, 3$. Let's compute the F. fn for $d=3$, following the method used for Laplace Grn. fn on pp 105-106

i) we transform to set $\vec{y} = 0$

ii) use sph coords in $\vec{k}$ -space $d^3 k = k^2 \sin \theta d\theta d\phi$

$$e^{i\vec{k} \cdot \vec{x}} = e^{i k r \cos \theta} \quad r = |\vec{x}|$$

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Then $G^{(3)}(\vec{x}, t) = \frac{1}{(2\pi)^3 c} 2\pi \int_0^\infty \frac{dk}{k} k^2 \sin(kt) \int_{-1}^1 d\cos\theta e^{ikr \cos\theta}$
 from $\int d\theta \rightarrow$ $\Delta = \frac{2}{kr} \sin kr$ as in Laplace case

$$= \frac{1}{2\pi^2 cr} \int_0^\infty dk \sin(kt) \sin(kr)$$

since integral is even

$$= \frac{1}{4\pi^2 cr} \int_{-\infty}^\infty dk \sin kr \sin kt$$

$$= \frac{1}{4} (e^{ikr} - e^{-ikr})(e^{ikt} - e^{-ikt})$$

$$= +\frac{1}{4} (e^{ik(r-t)} - e^{ik(r+t)} + c.c.)$$

Use std result $\frac{1}{2\pi} \int_{-\infty}^\infty dk e^{ikz} = \delta(z)$ (completeness of plane waves in 1-dim.)

Use this to do remaining $\int dk$ with result:

$$G^{(3)}(\vec{x}, t) = \frac{1}{4\pi cr} [\delta(r-ct) - \delta(r+ct)]$$

drop this since peak never occurs.

$$G^{(3)}(\vec{x}-\vec{y}, t) = \frac{1}{4\pi c |\vec{x}-\vec{y}|} \delta(|\vec{x}-\vec{y}| - ct)$$

Phys. interp: a wave emitted from pt source at $\vec{y}=0$ at $t=0$ is located on an expanding spherical shell of radius $r=ct$ at time t .

Amplitude of wave falls as $1/r$, because the wave energy is spread out over a shell of area $4\pi r^2$.

The same wave emitted from a general source $\frac{1}{c^2} g(\vec{y}, t)$ is

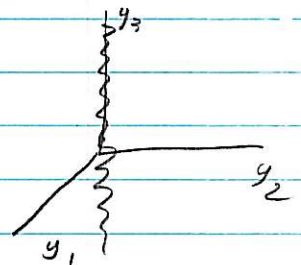
$$u(\vec{x}, t) = \frac{1}{4\pi c} \int_{-\infty}^t dt' \int d^3y \frac{\delta(|\vec{x}-\vec{y}| - c(t-t'))}{|\vec{x}-\vec{y}|} g(\vec{y}, t')$$

? change lower limit and put t' deep into source

Consider Prob 11.2, 12 on p 522 of H as an interesting example of 3 dim wave eqn with source.

Choose line source along y_3 axis which is instantaneous in t' :

$$g(\vec{y}, t') = \delta(y_1) \delta(y_2) \delta(t')$$



The source is trans. m.d. in y_3 , so integral (as we will show) is indep of x_3 . So the solution to prob. of instant, line source in $d=3$ will look like soln to instant point source in $d=2$ with coords x_1, x_2 . Then in

Hadamard's Method of descent: the source looks like instant pt source in the plane, so the result of the integral will be Green's fn $G^{(2)}(\vec{x}, t)$ for wave eqn in $d=2$:

Let's do it:

$$u(\vec{x}, t) = \frac{1}{4\pi c} \int_{-\infty}^{t'} dt' \int dy_1 dy_2 dy_3 \frac{\delta(|\vec{x}-\vec{y}| - c(t-t'))}{|\vec{x}-\vec{y}|} \delta(y_1) \delta(y_2) \delta(z')$$

The 4 δ fns will allow us to do all 4 integrals.

We can immediately integ. over z', y_1, y_2 to get

$$\begin{aligned} u(\vec{x}, t) &= \frac{1}{4\pi c} \int_{-\infty}^{\infty} dy_3 \frac{\delta(\sqrt{\vec{x}_\perp^2 + (x_3 - y_3)^2} - ct)}{\sqrt{\vec{x}_\perp^2 + y_3^2}} \quad \text{where } \vec{x}_\perp = (x_1, x_2) \\ &= \frac{1}{4\pi c} \int_{-\infty}^{\infty} dy \frac{\delta(\sqrt{\vec{x}_\perp^2 + y^2} - ct)}{\sqrt{\vec{x}_\perp^2 + y^2}} \quad \begin{array}{l} y = y_3 - x_3 \\ \text{indep of } x_3 \\ \text{is now clear.} \end{array} \end{aligned}$$

To do the remaining integral, we need change of variable rule

for δ fn: $\delta(f(s)) = \sum_i \frac{1}{|f'(s_i)|} \delta(s - s_i) \leftarrow \begin{array}{l} \text{sum over} \\ \text{the spikes} \\ \text{where } f(s) = 0 \end{array}$

In our case

$$\begin{aligned} f(y) &= \sqrt{\vec{x}_\perp^2 + y^2} - ct \\ f(y) = 0 &\quad \text{at } \vec{x}_\perp^2 + y^2 = c^2 t^2 \quad y_\pm = \pm \sqrt{c^2 t^2 - \vec{x}_\perp^2} \end{aligned}$$

The zeros occur only argum. of $\sqrt{\quad}$ is $\pm$ real; $|\vec{x}_\perp| < ct$.

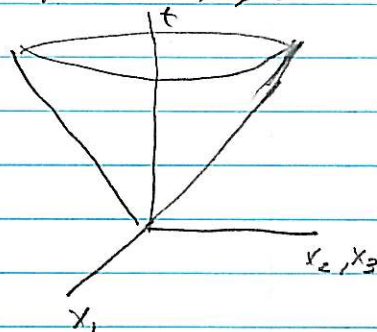
$$f'(y) = \frac{y}{\sqrt{\vec{x}_\perp^2 + y^2}}$$

Both $y_\pm$ contribute equally to the integral, Result

$$G^{(2)}(\vec{x}, t) = \begin{cases} 0 & \text{if } |\vec{x}| > ct \\ \frac{1}{2\pi c} \frac{1}{\sqrt{c^2 t^2 - \vec{x}_\perp^2}} & \text{if } |\vec{x}| < ct \end{cases}$$

Ex: Check explicitly that $(\frac{1}{c^2} \partial_t^2 - \nabla^2) G^{(2)}(\vec{x}, t) = 0$. It is harder to verify that the IC's are satisfied, so let's accept this

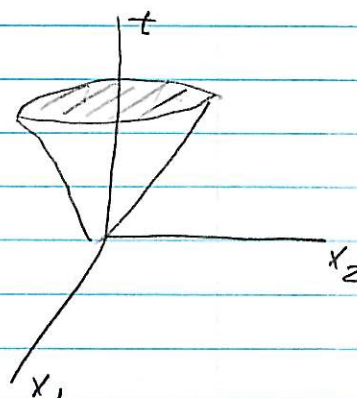
Compare wave prop in $d=3, 2$: Plot "forward light cone" from source at $\vec{y}=0, t=0$



For $d=3$, the response at time t is located on the surface of the cone because $G \sim \delta(|x| - ct)$

For $d=2$, the signal fills interior of cone, although intensity is peaked on surface, since

$$G \sim \frac{1}{\sqrt{c^2 t^2 - \vec{x}^2}}$$



General situation:

- 1) any odd dim $3, 5, 7, \dots$
response is on surface of cone.
Wave propagator with fixed speed $|\vec{v}| = c$.
- 2) any even dim $d=2, 4, 6$
response in interior of cone, wave propagator with spread of velocities $|\vec{v}| \leq c$.

Interesting application for education: space time dia of classroom with prof P at origin and student S at $\vec{x}$

In $d=3, 5, 7, \dots$ odd

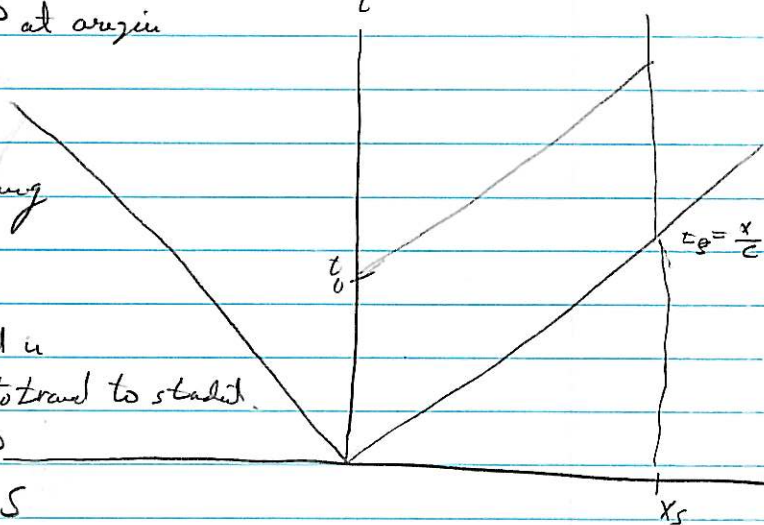
i) P speaks the truth beginning at $t=0$

ii) S hears the truth at time $t = t_s = x_s/c$, which is the time it takes for sound to travel to student.

iii) a later truth spoken by P

at time t_0 is heard by S

at time $t = x_s/c + t_0$. This agrees with idea of sound propagating at fixed speed c .



In $d = 2, 4, 6 \dots$ even.

Sound emitted by P moves with speed of velocities $v \leq c$.

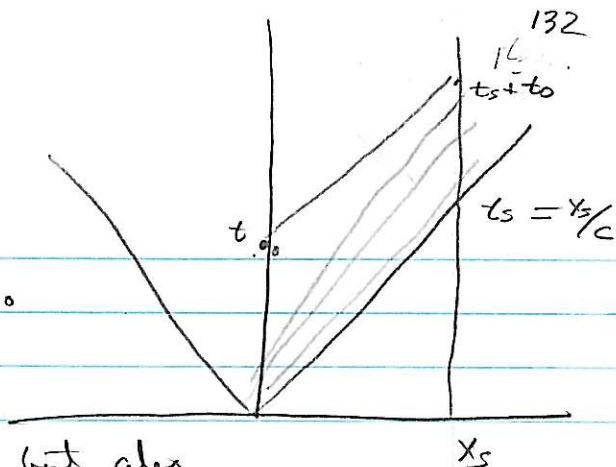
S hears the truth beginning at t_0 .

But at later time t_0 , S hears

not only the prompt wave of truth spoken by P at time t_0 , but also

a complex jumble of everything spoken at all earlier times $0 < t < t_0$.

Conclusion: Education would be difficult in even dim.



Th: Dec 10: Wave propagation in a medium with two layers

e.g. air + water. See Sec 4.6 of

Upper half of $\mathbb{R}^3$ $z > 0$ $c = c_+$

Lower half $z < 0$ $c = c_-$

PDE in each region. $\frac{1}{c_{\pm}^2} \partial_t^2 u_{\pm} = \nabla^2 u_{\pm}$

Sols are superpos. of plane waves

$$u_{\vec{k}_{\pm}}(\vec{x}, t) = e^{i(\vec{k}_{\pm} \cdot \vec{x} - \omega_{\pm} t)} \quad \text{with } |\vec{k}_{\pm}| = \omega_{\pm} / c_{\pm}$$

We will derive a picture in which we will have:

in region $z > 0$ Incident wave + Reflected wave
 $u_+ = e^{i(\vec{k}_I \cdot \vec{x} - \omega_I t)} + R e^{i(\vec{k}_R \cdot \vec{x} - \omega_R t)}$

in region $z < 0$: pure Transmitted wave

$$u_- = T e^{i(\vec{k}_T \cdot \vec{x} - \omega_T t)}$$

At the interface we will require that $u(x, y, z, t)$ and $\partial_z u(x, y, z, t)$ are continuous at $z = 0$.

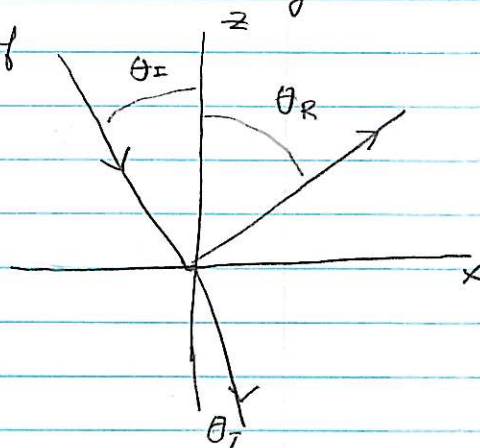
The technical problem is to find waves of the form u_+ , u_- above which satisfy these conds.

The problem has translation sym in the variables x, y —

So we expect to have waves of same freq. $\omega_I = \omega_R = \omega_T = \omega$
 and same x, y components of $\vec{k}_I, \vec{k}_R, \vec{k}_T$

The prob has rotation sym in x, y plane, so it's no loss of generality to take $k_y = 0$.

The most general set of wave vectors $\vec{k}$ which incorporate these conds are (with angles defined in drawing above)



$$\vec{k}_I = (k_+ \sin \theta_I, 0, -k_+ \cos \theta_I)$$

$$k_+ = \omega/c_+$$

$$\vec{k}_R = (k_+ \sin \theta_R, 0, k_+ \cos \theta_R)$$

$$\hookrightarrow k_+ \sin \theta_I$$

$$k_- = (k_- \sin \theta_T, 0, -k_- \cos \theta_T)$$

$$k_- = \omega/c_-$$

$$\hookrightarrow k_+ \sin \theta_I$$

bc of trans
sym in x.

From the "kinematics" we immediately derive
the well known laws of geom. optics!

$$1) \quad k_+ \sin \theta_I = k_+ \sin \theta_R \Rightarrow \theta_R = \theta_I$$

angle of reflection = angle of incidence

$$2) \quad k_- \sin \theta_T = k_+ \sin \theta_I, \text{ so } \frac{\sin \theta_T}{\sin \theta_I} = \frac{k_+}{k_-} = \frac{c_-}{c_+}$$

Snell's law of refraction: $\rightarrow$ look more closely

Rewrite $\sin \theta_T = \frac{c_-}{c_+} \sin \theta_I$

if $c_- < c_+$, then refracted wave is closer to vertical

if $c_+ < c_-$ refracted wave is farther from vertical

BUT there is a limit $\sin \theta_T \leq 1$ which occurs at

the critical value of the incident angle: $\sin \theta_c = c_+/c_-$

For $\theta_I > \theta_c$ our picture of a transmitted wave must be wrong.

So let's start by assuming $\sin \theta_I < \sin \theta_c = c_+/c_-$. Our kinematics

is then correct. Our sol's u_+, u_- satisfy PDE in each region.

We then look at interface conds.

$$a) \quad u_+(x, y, 0, t) = u_-(x, y, 0, t) \quad z=0$$

$$I + R = T$$

$$b) \quad \partial_z u_+(x, y, z=0, t) = \partial_z u_-(x, y, z=0, t)$$

use new notation $\vec{k}_I = (k_1, 0, -\tilde{k}_+)$ $k_R(k_1, 0, \tilde{k}_+)$

$$k_- = (k_1, 0, -\tilde{k}_-)$$

can drop common factor $e^{i(k_1 x - \omega t)}$

We still have from kinematics above:

$$k_1 = k_+ \sin \theta_I$$

$$\tilde{k}_+ = k_+ \cos \theta_I$$

$$\tilde{k}_- = k_- \cos \theta_T$$

The ∂_z continuity condition gives

$$-i\tilde{k}_+(1-R) = -i\tilde{k}_-T$$

$$(1-R) = \frac{\tilde{k}_-}{\tilde{k}_+} T$$

$$(1+R) = T$$

Add these to get

$$2 = \left(1 + \frac{\tilde{k}_-}{\tilde{k}_+}\right) T \Rightarrow$$

$$T = \frac{2\tilde{k}_+}{\tilde{k}_+ + \tilde{k}_-}$$

$$R = T - 1 = \frac{\tilde{k}_+ - \tilde{k}_-}{\tilde{k}_+ + \tilde{k}_-}$$

Ex: use our geom. relations above + Snell

+ trig identities to show that these R & T coeffs agree with text.

$$R = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)}$$

$$T = \frac{2 \sin \theta_T \cos \theta_I}{\sin(\theta_T + \theta_I)}$$

Other facts best seen from our formulae C:

1) T and R are indep of freq, ω of the wave:

2) $T^2 + R^2 = 1$ ← consequence of cons. of energy.

What happens when $\sin \theta_I > \sin \theta_c = c_+/c_-$

Our sol with transmitted oscillatory wave is not correct.

To see what happens look more carefully at

$$u_- = T e^{i(k_1 x - \tilde{k}_- z - \omega t)}$$

Then u a sol of wave eq if

$$k_1^2 + \tilde{k}_-^2 = \omega^2/c_-^2$$

$$\tilde{k}_-^2 = \frac{\omega^2}{c_-^2} - k_1^2 = \frac{\omega^2}{c_+^2} \left(\frac{c_+^2}{c_-^2} - \sin^2 \theta_I \right) < 0$$

$$\text{Then we write } \tilde{k}_- = +i|k_1| \sqrt{\sin^2 \theta_I - c_+^2/c_-^2} = i\gamma$$

$$\text{Then } u_- = e^{i(k_1 x - \omega t)} e^{-\gamma z}$$

← The wave intensity decays

exponentially as $z \rightarrow -\infty$

u called an evanescent wave:

We must again check the interface conditions:

$$1 + R = T \quad \text{as before}$$

$$-i\tilde{k}_+(1-R) = \gamma T$$

$$\text{Let } Q = \gamma/\tilde{k}_+$$

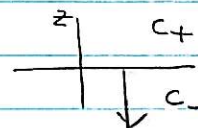
$$(1-R) = iQT$$

$$1Q(1+R) = iQT$$

subtract

$$1 - iQ = R(1 + iQ)$$

$$R = \frac{1 - iQ}{1 + iQ}$$



Note that $RR^* = 1$ (energy cons again)

$$\begin{aligned} \text{Then } R &= e^{-2i\delta} \quad \text{with } \delta = \tan^{-1} Q = \tan^{-1} \gamma/k_+ \\ &= \tan^{-1} \gamma/k_+ \cot \theta_I \\ &= \tan^{-1} \left(\frac{\gamma c_+}{\cot \theta_I \omega} \right) \end{aligned}$$

Summary: if $\sin \theta_I > c_+/c_-$;

then we have total reflection

of incoming wave. The reflected wave has same intensity as transmitted wave, but reflected wave is shifted in phase.

Last topic of 18.303/2009

Group velocity of waves:
Compare Ch 4.2 of H.

The wave eqn we have been studying:

$$\frac{1}{c^2} \partial_t^2 u = \nabla^2 u \quad \text{has plane wave sals. } e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

We had a homework problem on Sch. Eq.

$$i \partial_t \psi = -\frac{\nabla^2 \psi}{2m}$$

$$\text{which has plane wave } e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad \omega = c|\vec{k}|$$

Linearized KdV Eqn Sec 601 of H is a 1-dim eqn

$$\partial_t u = -c \partial_x u + \beta \partial_x^3 u \quad \text{has plane wave sals } e^{i(kx - \omega t)}$$

$$\omega = ck + \beta k^3$$

In general we want to study linear wave eqns which have plane wave sals:

$$e^{i(\vec{k} \cdot \vec{x} - \omega(\vec{k})t)}$$

We will not be specific about the eqn these plane waves come from. Instead we want to study the propagation of waves of this type and of superpositions which will also be sals of the linear eqns

$$u(\vec{x}, t) = \int d\vec{k} F(\vec{k}) e^{i(\vec{k} \cdot \vec{x} - \omega(\vec{k})t)} \quad \checkmark$$

$\omega(\vec{k})$ specifies how the frequency of a wave depends on the wave vector $\vec{k}$. Recall wavelength $\lambda = 2\pi/k$
 $\omega(\vec{k})$ specifies the dispersion relation of the waves in question. It is an important quantity.

$\vec{k} \cdot \vec{x}$ 

We also know that $\omega(k)$ tells us how the surface of constant phase $\vec{k} \cdot \vec{x} = \omega(k)t$ moves in time.

They move at velocity $\omega(k)\hat{k}$; so we say that ω is the phase velocity of the waves. However these plane waves which extend uniformly over space are an idealization which cannot be attained in practice. For example they would carry infinite energy which is impossible. It is more realistic to deal with wave packets or superpos. as we ✓

It is a quite general statement although it requires some approximation that wave packets travel with velocity $\vec{\nabla}_k \omega(\vec{k})$ which is called the group velocity.

Check dimensions $[\omega] = \frac{1}{t}$ for ω

$$[\vec{\nabla}_k \omega] = \frac{1}{1/l} \frac{1}{t} = \frac{l}{t} = \text{velocity}$$

Let's start with a superpos. which is a sum of two plane waves of nearby wave numbers k and $k+\Delta k$ (in one dim.)

$$u(x,t) = e^{i(kx - \omega(k)t)} + e^{i((k+\Delta k)x - \omega(k+\Delta k)t)}$$

↪ This is still unrealistic (so energy) but it will illustrate the idea

Assume $\omega(k+\Delta k) \approx \omega(k) + \Delta k \omega'$ ← linear approx in Taylor series.

Then

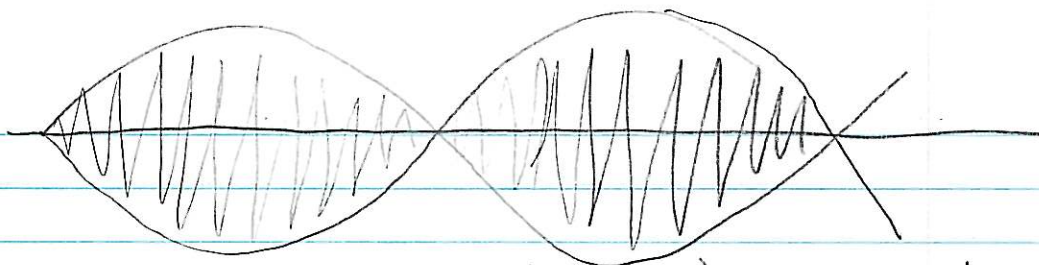
$$u(x,t) \approx e^{i(kx - \omega(k)t)} [1 + e^{i\Delta k(x - \omega'(k)t)}]$$

$$\text{use } 1 + e^{i\delta} = e^{i\delta/2} (e^{i\delta/2} + e^{-i\delta/2})$$

⇒

$$u(x,t) = 2e^{i\left[\left(k+\frac{\Delta k}{2}\right)x - \omega + \frac{1}{2}\omega'\Delta k t\right]} \cos\left[\frac{1}{2}\Delta k(x - \omega' t)\right]$$

Then a product of a plane wave of high wave number (short wavelength) $k + \frac{1}{2}\Delta k$ moving in an "envelope" formed by a wave of very small wave number Δk (and long wave length)



The short waves are moving with (essentially) the original phase velocity $\omega(k)$, but the envelope itself has argument $\sim x - \omega'(k)t$ so the envelope moves with the group velocity: In general we want to argue that physical effects such as energy transport move with group velocity.

We now consider the matter in terms of Gaussian Wave Packets. Recall basic FT of Gaussian from earlier in term:

$$e^{-\beta x^2} = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int_{-\infty}^{\infty} dk e^{-\frac{k^2}{4\beta}} e^{ikx}$$

Take slightly different $F(k) = \sqrt{\frac{\pi}{\beta}} e^{-\frac{(k-k_0)^2}{4\beta}}$
 peak amplitude at wave number $k=k_0$
 width of peak is $\Delta k = \sqrt{4\beta}$.

what is

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int dk e^{-\frac{(k-k_0)^2}{4\beta}} e^{ikx} \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} e^{ik_0 x} \int dk e^{-\frac{(k-k_0)^2}{4\beta}} e^{i(k-k_0)x} \\ &= e^{-\beta x^2} e^{ik_0 x} \quad \text{after change of variable } g \equiv k - k_0 \end{aligned}$$

Now let the Gaussian wave move as a sol of linear wave eqn with dispersion relation $\omega(k)$

$$u(x,t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int dk e^{-\frac{(k-k_0)^2}{4\beta}} e^{i(kx - \omega(k)t)}$$

Since $F(k)$ is sharply peaked around $k=k_0$, it should be reasonable to approximate $\omega(k) = \omega(k_0) + (k-k_0)\omega'(k_0)$

$$\begin{aligned} \text{Then} \quad u(x,t) &= \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} e^{i(k_0 x - \omega_0 t)} \int dk e^{-\frac{(k-k_0)^2}{4\beta}} e^{i(k-k_0)(x - \omega'_0 t)} \\ &= \omega_0 + (k-k_0)\omega'_0 \end{aligned}$$

The integral is now easily done with some var. $g = k - k_0$,
and we get:

$$u(x,t) = e^{i(k_0 x - \omega_0 t)} e^{-\beta(x - \omega'_0 t)^2}$$

Again a plane wave with usual phase velocity inside a
Gaussian envelope which moves with group velocity.

It should be a more accurate approx to take

$$\omega(k) = \omega_0 + (k - k_0) \omega'_0 + \frac{1}{2} (k - k_0)^2 \omega''_0$$

The integral is then

$$u(x,t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} e^{i(k_0 x - \omega_0 t)} \int dg e^{-\frac{g^2}{4\beta}} e^{-i\omega''_0 \frac{g^2 t}{2}} e^{i g(x - \omega'_0 t)}$$

This is very similar to evolution of Gaussian wave packet
under heat flow on p 89 of note

$$u = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int dk e^{-\frac{k^2}{4\beta}} e^{-k^2 t} e^{i k x}$$

$$= \frac{1}{\sqrt{1 + 4\beta t}} e^{-\beta x^2 / (1 + 4\beta t)}$$

We can use this result if we relate $x \rightarrow x - \omega'_0 t$ replace w
group v

$$t \rightarrow \frac{i\omega''_0}{2} t \equiv i\alpha t$$

The result in our case is then

$$u(x,t) = e^{i(k_0 x - \omega_0 t)} \frac{1}{\sqrt{1 + 4i\beta\alpha t}} e^{-\frac{\beta(x - \omega'_0 t)^2}{(1 + 4i\beta\alpha t)}}$$

The ^{new} main effect is that the wave packet spreads as t increases

$$1 + 4i\beta\alpha t = \sqrt{1 + 16\beta^2\alpha^2 t^2} e^{i\theta_t} \quad \tan \theta_t = 4\beta\alpha t$$

The width is $\left(\frac{\sqrt{1 + 16\beta^2\alpha^2 t^2}}{\beta} \right)^{1/2}$ $\leftarrow$ shows spreading

See Merzbacher Quantum Mechanics ch 8.

See Hakenman Sec 4.4 for an interesting discussion
of fiber optic wave propagation.