

Tues Dec 1:

Wrap up our work on Laplace/Poisson Green's function

1. Universal short distance behaviour of $G(\vec{x}, \vec{y})$ in finite regions

2. A relation between $G(\vec{x}, \vec{y})$ and $G(\vec{x}, \vec{y}, \pm)$ for heat eq

3. $G(\vec{x}, \vec{y})$ for interior of sphere (a ball)

... method of images

Q. Find a brief summary of all prev. results for $G(\vec{x}, \vec{y})$

i) For finite regions we have spectral rep.

$$G_D(\vec{x}, \vec{y}) = - \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{\lambda_n N_n^2} \quad \text{true in any } d$$

ϕ^D or ϕ^N

ii) For full $\mathbb{R}^d$ we have explicit form

$$G^{(d)}(\vec{x}, \vec{y}) = \frac{1}{(d-2)\Omega_d} \frac{1}{|\vec{x}-\vec{y}|^{d-2}} \quad \text{or} \quad \frac{1}{2\pi} \ln\left(\frac{|\vec{x}-\vec{y}|}{r_0}\right) \quad d=2$$

iii) image charge methods are powerful (and fun) when they work

1. Claim: $G_{D,N}$ for any finite R in d dims has the property

$$G_{D,N}(\vec{x}, \vec{y}) = G^{(d)}(\vec{x}, \vec{y}) + V(\vec{x}, \vec{y}) \quad \text{where} \quad \text{Sec 4.5, 7 of H.}$$

$V(\vec{x}, \vec{y})$ is regular at $\vec{x} = \vec{y}$ (and in all of R)

Significance is that short distance behaviour is universal

same for finite regions R of any shape and any BC's

for fixed d . Short dist. behaviour is same as $G^{(d)}$ for infinite space $\mathbb{R}^d$.

Pf of this beautiful fact for D prob on V with bdy S in $d \geq 3$

$$G_D(\vec{x}, \vec{y}) \text{ satisfies } \nabla_V^2 G_D(\vec{x}, \vec{y}) = \delta^d(\vec{x}-\vec{y}) \quad G_D(\vec{x}, \vec{y}) \Big|_{\vec{x} \in S} = 0$$

Then $V(\vec{x}, \vec{y}) = G_D - G^{(d)}$ must satisfy

$$\nabla_V^2 V(\vec{x}, \vec{y}) = 0$$

$$V(\vec{x}, \vec{y}) \Big|_{\vec{x} \in S} = -G^{(d)} = \frac{1}{(d-2)\Omega_d} \frac{1}{|\vec{x}-\vec{y}|^{d-2}} \Big|_{\vec{x} \in S}$$

There is a bdy value prob for Laplace with specific bdy data.

There is a PDE in variable $\vec{x}$ with $\vec{y}$ just a spectral: $\vec{y} \in V$

We already know quite a bit about these prob. Let's review

- i) Sol. is unique as was proved by Green's 1st identity on p 73
- ii) The bdy data are smooth as $\vec{x}$ varies over $S = \partial V$
(true since $\vec{y}$ is an interior pt, so $|\vec{x} - \vec{y}|$ never vanishes.)
For smooth bdy data, the sol of bdy value prob. is also smooth. (We saw this in an early example of Lap/D prob on $\bigcirc$ where we derived Poisson integral formula).
- iii) The smooth sol for $V(\vec{x}, \vec{y})$ is smooth at all interior pts including $\vec{x} = \vec{y}$. So our pf is complete.

2. Analogy between Green's fu for heat flows and Laplace.
(background info for prob 26 of PSat 6)

- i) Recall our discussion on p 96.5 of Green sol to problem of Heat flow on $\mathcal{R}$ with D bdy data

$$\partial_t u(\vec{x}, t) = \nabla^2 u(\vec{x}, t) \quad u(\vec{x}, t)|_{\partial \mathcal{R}} = A(\vec{x}, t) \text{ given}$$

We solved the prob for 1dim case, we also have the following formula of some structure for general d: (actually for $d=2$)

$$u(\vec{x}, t) = - \int_0^t dt' \int_{\partial \mathcal{R}} d\vec{y} \cdot \vec{n} \cdot \vec{\nabla}_{\vec{y}} G_0(\vec{x}, \vec{y}, t-t') \Big|_{\vec{y} \in \partial \mathcal{R}} A(\vec{y}, t') \quad *$$

$$\text{Spectral rep} \quad G_0(\vec{x}, \vec{y}, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} \quad \phi_n^0(\vec{x})$$

- ii) on p 76 we used sol of Lap/D prob with bdy data $f(\vec{x})$.

$$u(\vec{x}) = \nabla^2 u(\vec{x}) = 0 \quad \vec{x} \in \mathcal{R} \quad u(\vec{x})|_{\partial \mathcal{R}} = f(\vec{x})$$

$$u(\vec{x}) = \int_{\partial \mathcal{R}} d\vec{y} \cdot \vec{n} \cdot \vec{\nabla}_{\vec{y}} G_0(\vec{x}, \vec{y}) \Big|_{\vec{y} \in \partial \mathcal{R}} f(\vec{y}) \quad **$$

- iii) We introduced Laplace on p 54 by physical reasoning concerning long time limit of heat flow. Specifically we argued that

$$u(\vec{x}, t) \xrightarrow{t \rightarrow \infty} u(\vec{x})$$

with $u|_{\partial \mathcal{R}} = f(\vec{x}) \quad \xrightarrow{t \rightarrow \infty} \quad 0 \text{ sol of Lap with } u(\vec{x})|_{\partial \mathcal{R}} = f(\vec{x})$

This suggests a connection between heat flows and Laplace prob.
at least for t - indep bdy data $A(\vec{y}, t) = f(\vec{y})$

In this case we can exclude order of integ in $\mathbb{R}^3$ and write

$$u(\vec{x}, t) = - \int_{\partial \mathbb{R}^3} dy f(\vec{y}) \vec{n} \cdot \vec{\nabla}_{\vec{y}} \int_0^t dt' G_D(\vec{x}, \vec{y}, t-t')$$

Compare with $\mathbb{R}^3$, we should have $\lim_{t \rightarrow \infty} u(\vec{x}, t) = u(\vec{x})$

for all choices of bdy data $f(\vec{y})$. This suggests a known relation betw. the Green's fns for heat flow and Lap.

$$\lim_{t \rightarrow \infty} - \int_0^t dt' G_D(\vec{x}, \vec{y}, t-t') = G_D(\vec{x}, \vec{y})$$

Let's prove this relation using spectral repn:

$$G_D(\vec{x}, \vec{y}, t-t') = \sum_{n=1}^{\infty} e^{-\lambda_n(t-t')} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2}$$

We need only the elementary integral

$$\int_0^t dt' e^{-\lambda_n(t-t')} = \frac{1}{\lambda_n} e^{-\lambda_n(t-t')} \Big|_0^t$$

$$= \frac{1}{\lambda_n} (1 - e^{-\lambda_n t})$$

$$\xrightarrow{t \rightarrow \infty} 1/\lambda_n$$

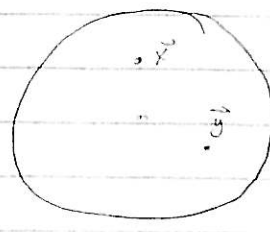
Then

$$\lim_{t \rightarrow \infty} - \int_0^t dt' G_D(\vec{x}, \vec{y}, t-t') = - \sum \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{\lambda_n N_n^2} = G_D(\vec{x})$$

In Prob 26 you are asked to verify this for Green's fn on full $\mathbb{R}^3$.

3. Green's fn in a barrel ball in $\mathbb{R}^3$, a non-trivial application of image charge methods. See Sec 9.5.9 of H for case of disc.

Prob find soln: $\nabla_{\vec{x}}^2 G_D(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}) \quad |\vec{x}| \leq a$
 $G_D(\vec{x}, \vec{y}) \Big|_{|\vec{x}|=a} = 0$



It is quite remarkable that there is closed soln

in unwrapping full space $G^{(3)} = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}$ with direct source + one image

$$G_D(\vec{x}, \vec{y}) \stackrel{?}{=} -\frac{1}{4\pi} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x} - \vec{y}_I|} \right]$$

a) This will satisfy PDE as long as $|\vec{y}_I| > a$ (image outside)

b) How do we choose $\vec{y}_I$ to satisfy BC $G_D(\vec{x}, \vec{y}) \Big|_{\vec{x}=a\hat{x}} = 0$

c) Symmetry suggests $\vec{y}_I$ should be in same line through center $a\hat{x}$ (in order not to introduce a new direction which would lose symmetry).

(i) So we write $\vec{y}_I = f(y) \vec{y}$ $\vec{y} = |\vec{y}| \hat{y}$

(ii) Correct choice is suggested by idea of inversion

$$\vec{y}_I = \frac{a^2 \vec{y}}{y^2} \quad |\vec{y}_I| = \frac{a^2}{y} \quad |\vec{y}_I| |\vec{y}| = a^2$$

$$\text{as } \vec{y} \rightarrow 0 \quad \vec{y}_I \rightarrow \infty$$

$$\text{as } \vec{y} \rightarrow a\hat{x} \quad \vec{y}_I \rightarrow a\hat{x}$$

direct source and image source
 $\rightarrow$ same pt in body as in
 our prev. cases of
 images

So we can try

$$G_D = -\frac{1}{4\pi} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x} - \frac{a^2 \vec{y}}{y^2}|} \right]$$

$$\text{But as } y \rightarrow 0 \quad y_I \rightarrow \infty \quad \text{so } G_D \rightarrow -\frac{1}{4\pi} \left[\frac{1}{|\vec{x}|} - 0 \right]$$

So we cannot satisfy BC. Our ansatz needs further modification $G = -\frac{1}{4\pi} \left[\frac{1}{|\vec{x} - \vec{y}|} - h(y) \frac{1}{|\vec{x} - \vec{y}_I|} \right] \leftarrow$ this still satisfies PDE in $\vec{x}$. With $h(y) = \frac{a}{y}$ can satisfy BC.

The result can be written as

$$G_D(\vec{x}, \vec{y}) = -\frac{1}{4\pi} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{a}{y} \frac{1}{|\vec{x} - \vec{y}_I|} \right]$$

Note that this satisfies the "universality" property. The image term is regular for $|\vec{x}| \leq a$. To show that it satisfies the BC requires some relations in Sec 9.5.9 of H (which I will not give to save time)

Using these one can show that

$$G_D(\vec{x}, \vec{y}) = -\frac{1}{4\pi} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{a}{(x^2 y^2 - 2a^2 \vec{x} \cdot \vec{y} + a^2)^{1/2}} \right]$$

Ex: This satisfies reciprocity $G_D(\vec{x}, \vec{y}) = G_D(\vec{y}, \vec{x})$

Ex: Show that $G_0(\vec{x}, \vec{y}) \Big|_{\vec{x} = a\vec{x}} = 0 \leftarrow \text{BC}$

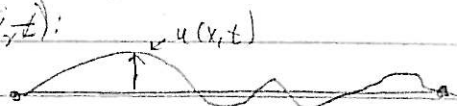
4. The Wave Eqn: (See Haberman Ch 4, 11, 12)

(i) Let $Q(\vec{x}, t)$ denote pressure in atmosphere at pt $\vec{x}$ and time t .

Then the PDE which describes propagating sound waves is

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} Q(\vec{x}, t) = \nabla^2 Q(\vec{x}, t)$$

(ii) Consider a string stretched betw. two supports, and displaced vertically from its rest position by $u(x, t)$:



The string responds to the initial stretch with motion governed by

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)$$

In both cases $[c] = \ell/t$. So c is a velocity, the speed of sound waves in the air or along string.

In both cases the physical mechanism involves

$\frac{\partial^2}{\partial t^2} Q$ or $\frac{\partial^2}{\partial t^2} u$ local acceleration of the medium

$\nabla^2 Q$ or $\frac{\partial^2}{\partial x^2} u$ "stretching" or "compression" of the medium due to internal forces.

It is difficult to give a good derivation of the PDE from a realistic model of internal forces. (See Sec 4.2 of H.). So we will assume the PDE's above and try to understand the associated mathematics and its physical consequences.

First let's discuss the extra conds. needed to get unique sol:

a) IC's on PDE is 2nd order in time t , so we must

specify two IC's: initial position $Q(\vec{x}, 0) = f(x)$

" velocity $\partial_t Q(\vec{x}, 0) = g(x)$

b) If $\exists$ a bdy, we must specify Dir N BC's, which describe physical conds. at bdy.

As one example, consider stretched drumhead over region R in plane
with $\partial R = C$



The vertical vibrations of drumhead obey wave eqn

PDE $\frac{1}{c^2} \partial_t^2 u(\vec{x}, t) = \nabla^2 u(\vec{x}, t)$ with

BC $u(\vec{x}, t)|_C = 0 \Rightarrow \partial_t u(\vec{x}, t)|_C = 0$

Given the configuration of the drumhead, we define the functional

$$E[u]_t = \frac{1}{2} \int_R d^2x \left[\frac{1}{c^2} (\partial_t u)^2 + \vec{\nabla} u \cdot \vec{\nabla} u \right] \geq 0$$

↑ This is called a functional, because it depends on values of $u(\vec{x}, t)$ throughout R . Given $u(\vec{x}, t)$, $E[u]_t$ is an ordinary function of t .

Physical Interpretation $E[u]_t$ is energy stored in oscillatory motion of drumhead. Motivation for this claim:

- i) $E[u]_t$ is sum of kinetic energy $(\partial_t u)^2$ + potential energy of stretching.
- ii) $E[u]_t$ is conserved in t if PDE + BC are satisfied.

$$\begin{aligned} \frac{d}{dt} E[u]_t &= \int_R d^2x \left[\frac{1}{c^2} (\partial_t u) (\partial_t^2 u) + \vec{\nabla} u \cdot \partial_t \vec{\nabla} u \right] \\ &= \int_R d^2x \left[(\partial_t u) \left(\frac{1}{c^2} \partial_t^2 u - \nabla^2 u \right) + \underbrace{\partial_t u \nabla^2 u + (\vec{\nabla} \partial_t u) \cdot \vec{\nabla} u}_{\vec{\nabla} \cdot (\partial_t u \vec{\nabla} u)} \right] \end{aligned}$$

/ Gauss Law

$$= \int_C dx \underbrace{\partial_t u}_{0 \text{ by BC}} \underbrace{\vec{\nabla} \cdot \vec{\nabla} u}_{\uparrow \text{ For Neu. BC this would vanish}}$$

For either Dir N, $\frac{d}{dt} E[u]_t = 0$ if PDE + BC are satisfied.

So $E[u]_t = E[u]_0 = \frac{1}{2} \int d^2x \left[\frac{1}{c^2} g(\vec{x})^2 + \nabla f \cdot \nabla f \right]$

Energy is fixed by initial conditions.

We will use the energy functional to obtain a simple proof of uniqueness.

Suppose $u_1(\vec{x}, t)$ and $u_2(\vec{x}, t)$ both satisfy all our conditions, namely PDE, IC's and BC. Then

$v(\vec{x}, t) \equiv u_1(\vec{x}, t) - u_2(\vec{x}, t)$ satisfies PDE (by linearity),
+ trivial BC + trivial IC's

Then $E[V]_{t=0} = 0$, but conservation means that

$$E[V]_t = 0 = \frac{1}{2} \int d^d x \left[\frac{1}{c^2} (\partial_t V)^2 + \vec{\nabla} V \cdot \vec{\nabla} V \right] \geq 0$$

But this non-negative quantity can vanish only if $V(\vec{x}, t) = \text{const.}$

The IC then forces $V(\vec{x}, t) \equiv 0$ for all $\vec{x}, t$ and we have proven uniqueness.

Summary: the wave eqn with reasonable BC's and IC's leads to a well posed problem with unique sol. The energy functional was very useful in our analysis.

General comments

- i) for $d \geq 2$, one can study the wave eqn by our tried and true methods of sep. of vars, eigenfn expansion $\Rightarrow$ Green's fn.
- ii) for $d=1$, those methods also work, but there are special techniques available which lead quickly to exact sol + some physical insight

So let's start our work on 1-dim case: $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)$

Think only about PDE, temporarily ignore BC + IC.

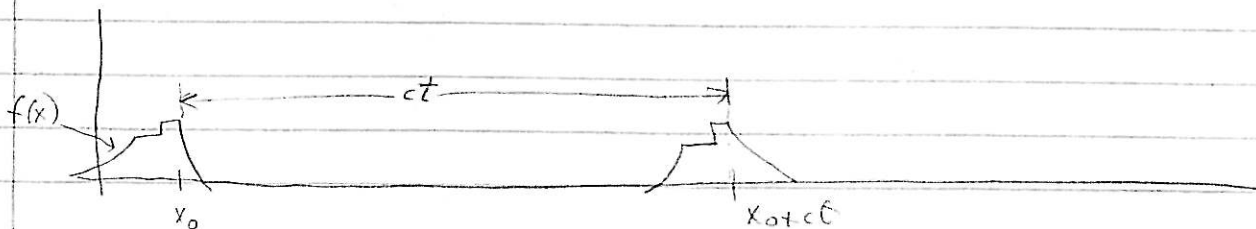
Observe: an arb. fn $f(x-ct)$ satisfies the PDE.

pf by chain rule:

$$\begin{aligned} \partial_x f &= f'(x-ct) & \partial_x^2 f &= f''(x-ct) \\ \partial_t f &= -cf'(x-ct) & \partial_t^2 f &= c^2 f''(x-ct) \end{aligned}$$

So $\frac{1}{c^2} \partial_t^2 f = \partial_x^2 f$.

This sol describes a wave (i.e. a disturbance of the medium) which moves rigidly to right with velocity c .



A initial disturbance with sharp leading edge at x_0 at $t=0$, has leading edge at $x=x_0+ct$ at time t . Every detail of the initial shape is preserved at later time t .

Contrast this with one dim heat eq in which an initial temp dist with sharp structure is immediately smoothed at small t

