

There is a correct formula for G_p because it has some properties

i) $(\partial_t - \partial_x^2) G_p(x-y, t) = 0$

ii) PBC $G_p(\pi-y, t) = G_p(-\pi-y, t) \quad \checkmark$

$$\partial_x G_p(\pi-y, t) = \partial_x G_p(-\pi-y, t)$$

iii) future evoln of pt source at $x=y$, (plus periodically spaced images)

iv) symmetry $G_p(x-y, t) = G_p(y-x, t)$

Prove $\checkmark$:

$$G_p(\pi-y, t) \sim \sum_{m=-\infty}^{\infty} e^{-(y+(2m-1)\pi)^2/4t} \quad \leftarrow \text{term with } m=1, 2, 3, \dots$$

in top sum is exactly

$$G_p(-\pi-y, t) \sim \sum_{m'=-\infty}^{\infty} e^{-(y+(2m'+1)\pi)^2/4t}$$

equal to term with $m'=0, 1, 2, \dots$

Rate of convergence: $\left\{ \begin{array}{l} \text{best at small } t \text{ when only a few image} \\ \text{terms have appreciable size} \\ \text{Still reasonable at large } t. \end{array} \right.$

Summary: image charge expansion of G_p is correct, and has complementary convergence properties to spectral rep.

Heat flow on $[0, \pi]$: B.C.'s: $\begin{cases} D & \text{at } x=0 \text{ and } x=\pi \\ N & \text{at } x=0 \text{ and } x=\pi \end{cases}$

Try $G_{D,N}(x, y, t) = G_p(x-y, t) \mp G_p(x+y, t)$

It satisfies $(\partial_t - \partial_x^2) G = 0$

It satisfies $\begin{matrix} D \\ N \end{matrix}$ BC at $x=0$ by construction

(It is odd fn of y by inspection. By symmetry

$$G_{D,N}(x, y, t) \equiv G_p(y-x) \mp G_p(y+x) \quad \leftarrow \text{now odd/even in } x \text{ by inspection.}$$

What about $x=\pi$

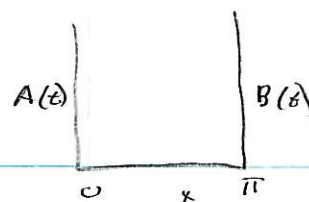
$$\begin{aligned} G_{D,N}(\pi, y, t) &= G_p(\pi-y, t) \mp G_p(\pi+y, t) \\ &= G_p(y-\pi, t) \mp G_p(y+\pi, t) \end{aligned}$$

D comb = 0 by periodicity of G_p

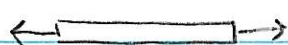
So we do get a new image charge rep for $G_D(x, y, t)$.

Ex: does $\partial_x G_N(x, y, t) \Big|_{x=\pi} = 0$?

Last applic of heat eg: Go back to our result for inhom. Dirichlet problem



$$u(x, t) = - \int_0^t dt' \left[\frac{\partial}{\partial y} G(x, y, t-t') \Big|_{y=\pi} B(t') \right.$$

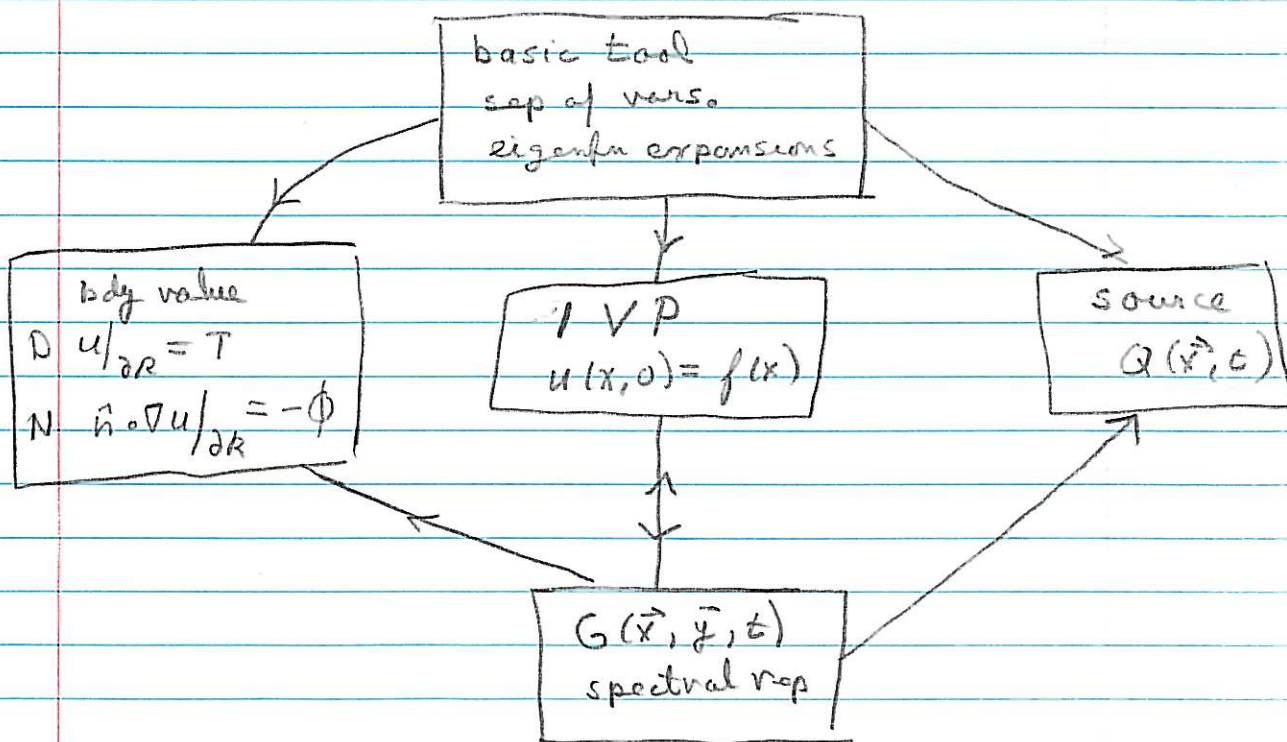


$$- \frac{\partial}{\partial y} G(x, y, t) \Big|_{y=0} A(t') \Big]$$

Can we simply write a formula of same structure to solve the corresp. prob in 2 dims.

See p 96 b

General flow chart for our work on heat flow in all situations:



Then $\nabla_x^2 G_0(\vec{x}, \vec{y}) = \sum_n \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} = \delta(\vec{x} - \vec{y})$

Using G_D , we can write the unique sol of Dirichlet with both source + bdy data

$$u(\vec{x}) = \int_{\mathcal{R}} d^2y G_D(\vec{x}, \vec{y}) \rho(\vec{y}) + \int_{\partial\mathcal{R}} d\gamma \hat{n} \cdot \nabla_y G_D(\vec{x}, \vec{y}) f(\vec{y})$$

We would like to find G_N for Neumann prob, but $\exists$ trouble due to 0-mode $\phi_0(\vec{x}) = 1/A$ $A = \text{area of } \mathcal{R}$

We might try $G_N(\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2}$

where $\phi_n(\vec{x})$ are Neumann eigenfns $\nabla^2 \phi_n = -\lambda_n \phi_n$
 $\hat{n} \cdot \vec{\nabla} \phi_n|_{\partial\mathcal{R}} = 0$

BUT zero mode term is ∞

There are several ways to define a modified Green's fn to avoid this problem. One way is to define the Neumann fn by.

$$G_N(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2} \quad \leftarrow n=0 \text{ mode is excluded so sum is well defined.}$$

BUT $\nabla_x^2 G_N = \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} = \delta(\vec{x} - \vec{y}) - \frac{1}{A}$

Correction to rhs. b.c. completeness sum includes $\phi_0 = 1/A$

We now show that the G_N allows us to solve

Neumann / Poisson prob $\nabla^2 u = \rho(\vec{x}) \quad \vec{x} \in \mathcal{R} \quad \hat{n} \cdot \vec{\nabla} u|_{\partial\mathcal{R}} = 0$

We would try: $u(\vec{x}) = \iint_{\mathcal{R}} d^2y G_N(\vec{x}, \vec{y}) \rho(\vec{y})$

$$\nabla^2 u = \iint_{\mathcal{R}} d^2y [\delta(\vec{x} - \vec{y}) - \frac{1}{A}] \rho(\vec{y}) = \rho(\vec{x}) - \frac{1}{A} \iint_{\mathcal{R}} d^2y \rho(\vec{y}) = 0$$

2nd term vanishes because of solvability cond, in prob pose

so our sol with modified Green's fn works!

Poisson's Eqn in $\mathbb{R}^3$: $\nabla^2 u(x) = \rho(x)$

(The body cond in finite R is replaced by a g.wth cond. as $|x| \rightarrow \infty$, but we won't be explicit about this.)

We expect Green sol of form $u(x) = \int d^3y G(x-y) \rho(y)$

For G we will write a spectral rep. analogous to * on p 103, but using complete set of plane waves.

$$G(\vec{x}-\vec{y}) = \frac{-1}{(2\pi)^3} \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{y}}}{k^2} \quad \phi_{\vec{k}}(\vec{x}) \equiv \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$$

i) G is a fn of $\vec{x}-\vec{y}$ because of trans. sym of $\mathbb{R}^3$

ii) We will see that integral is well defined despite $1/k^2 \rightarrow \infty$ as $k \rightarrow 0$.

iii) formal check of PDE

$$\nabla_x^2 G(\vec{x}-\vec{y}) = -\frac{1}{(2\pi)^3} \int d^3k \frac{-k^2 e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{y}}}{k^2} = \delta(\vec{x}-\vec{y})$$

completeness

Calculate G explicitly - no loss of generality if we take $y=0$

$$G(\vec{x}) = -\frac{1}{(2\pi)^3} \int \frac{d^3k}{k^2} e^{i\vec{k}\cdot\vec{x}}$$

The integral is best done in sph. coords in k -space

$$k_3 = k \cos \theta$$

$$k = (k_1^2 + k_2^2 + k_3^2)^{1/2}$$

$$k_1 = k \sin \theta \cos \phi$$

$$k_2 = k \sin \theta \sin \phi$$

$$d^3k = k^2 dk \sin \theta d\theta d\phi$$

$$\vec{k} \cdot \vec{x} = k|x| \cos \theta \leftarrow \text{convenient to}$$

choose coords so that geom. angle in

$k \cdot x$ is equal to $\cos \theta$ of sph. coords.

We then get:

$$G(\vec{x}) = \frac{-1}{(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2} \int_0^\pi d\theta \sin \theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} e^{i k |x| \cos \theta}$$

a) we $\int_0^{2\pi} d\phi = 2\pi$ $z = \cos \theta$

b) $\int_0^\pi d\theta \sin \theta e^{i k |x| \cos \theta} = \int_{-1}^1 dz e^{i k |x| z} = \frac{1}{i k |x|} (e^{i k |x|} - e^{-i k |x|}) =$

