

Similarly an arb fn $f(x+ct)$ is a sol. of wave eqn which describes a disturbance rigidly moving to the left with $v = -ct$.

The following gives some insight into why this simplification occurs.

The wave operator in 1-dim (only!) factors

$$(\partial_t^2 - c^2 \partial_x^2)u = (\partial_t - c \partial_x)(\partial_t + c \partial_x)u = (\partial_t + c \partial_x)(\partial_t - c \partial_x)u$$

1st factored form shows that any sol of 1st order $(\partial_t + c \partial_x)u = 0$ is also a sol of original second order problem.

It should be clear that the general sol of $(\partial_t + c \partial_x)u = 0$

is $u(x,t) = f(x-ct) \leftarrow f(z) \text{ is arbitrary.}$

Analogous remarks hold for the 2nd factored form.

This discussion leads to method of characteristics developed in Secs 12.2 and 12.3 of H. We will use this info in a somewhat different way.

1. Since the 2nd order wave eqn is linear, any superp. of right- and left-moving waves

$$u(x,t) = f_R(x-ct) + f_L(x+ct)$$

This is a sol which contains two arbitrary fns. But we know that the wave eqn has a unique sol making the two arb. fns in initial data: $u(x,0) = f(x)$ $\partial_t u(x,0) = g(x)$

This suggests that $\checkmark$ is the general sol., and we now show this ^{is true} for the full line $\mathbb{R}$, where there are no bdy's and no BC's.

1. Look at $u(x,t) = \frac{1}{2} [f(x-ct) + f(x+ct)]$

This satisfies the PDE + IC. $u(x,0) = f(x)$

$$\partial_t u(x,0) = 0$$

So we have solved the IVP for arbit. initial position data and zero velocity data.

2. To incorporate velocity data the obvious thing to try is

$$v(x,t) = \frac{1}{2c} [h(x+ct) - h(x-ct)]$$

This satisfies PDE and IC. $v(x,0) = 0$ and

$$\partial_t v(x,0) = h'(x), \text{ so we take } h'(x) = g(x)$$

$\swarrow$ initial vel. data

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Thus ~~the~~ $v(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} dy g(y) = \frac{1}{2c} [h(x+ct) - h(x-ct)]$

Then the ~~is~~ unique sol for arb. initial pos. + vel. data in

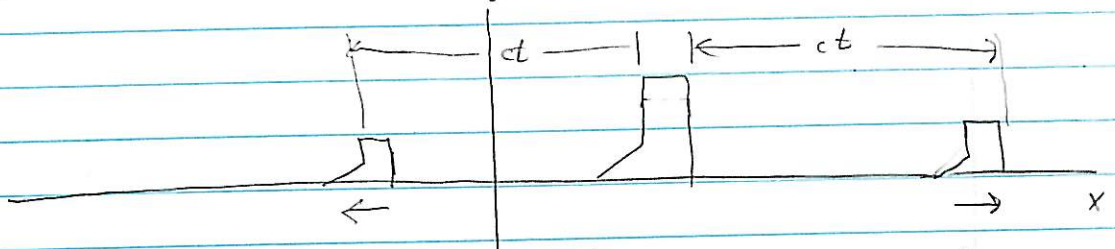
$$u(x,t) = \frac{1}{2} \left[f(x+ct) + f(x-ct) + \frac{1}{c} \int_{x-ct}^{x+ct} dy g(y) \right]$$

This is D'Alembert's sol of 1-dim wave eqn on $\mathbb{R}$. Sec 12.3.3 of H
(D'Alembert lived 1717-1783.)

Explore properties of this sol:

- i) Supp. $g(x) = 0$. The sol with general pos. data is
- $$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

The initial disturbance (of medium) splits into two $\frac{1}{2}$ -strength copies, one moves rigidly to right, the other to left.



Emphasize (again) rigid propagation. The propagating waves are exact ($\frac{1}{2}$ -str.) copies of initial $f(x)$. (Different from heat flow!)
(true only in $d=1$)

- ii) The g -part of sol best illustrates causal properties of wave prop. What regions in x or y of initial data affect value of $v(x,t)$ at time t ?

Spacetime diag. illustrates this $\rightarrow$

We have drawn "backward light cone"

from pt (x,t) . This intersects $t=0$ line

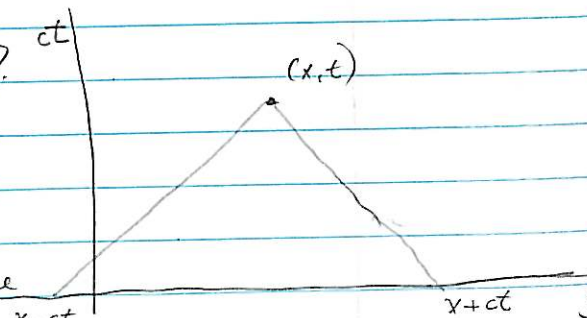
at pts $x_{\pm} = x \pm ct$. (If vertical

scale in ct , the cone has slope $\pm 45^\circ$). The interval

$x_- < y < x_+$ is called "domain of dependence" of the pt (x,t) .

Initial data inside this domain determines the value of wave at (x,t) .

" " outside " " has no influence.



Consider a $\frac{1}{2}$ -line problem in order to understand effect of a bdy.

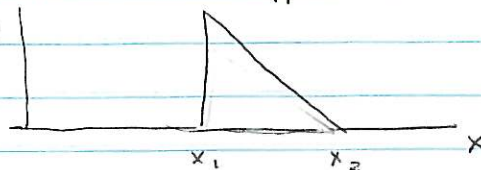
PDE $\frac{1}{c^2} \partial_t^2 u = \partial_x^2 u \quad 0 < x < \infty$

IC. $u(x, 0) = f(x) \quad \partial_t u(x, 0) = 0 \leftarrow \text{for simplicity}$

CD $u(0, t) = 0 \quad (\text{Dirichlet})$

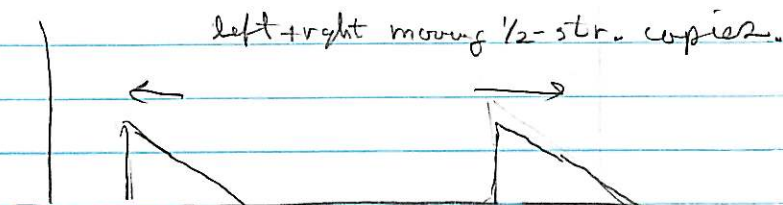
It should be reasonably clear wave happens to initial disturbance $f(x)$

$t=0$



- i) body has no effect until time $t_1 = x_1/c$ where left edge of initial $f(x)$ reaches $x=0$.

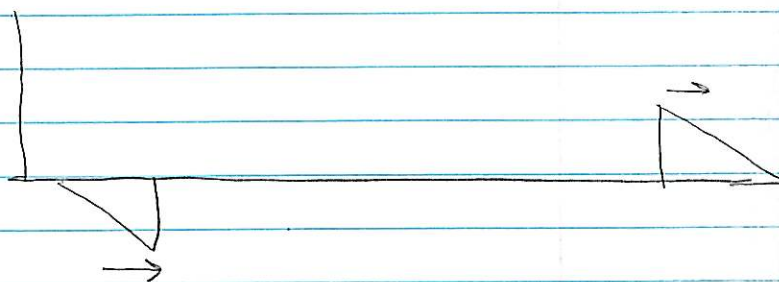
$0 < t < t_1$



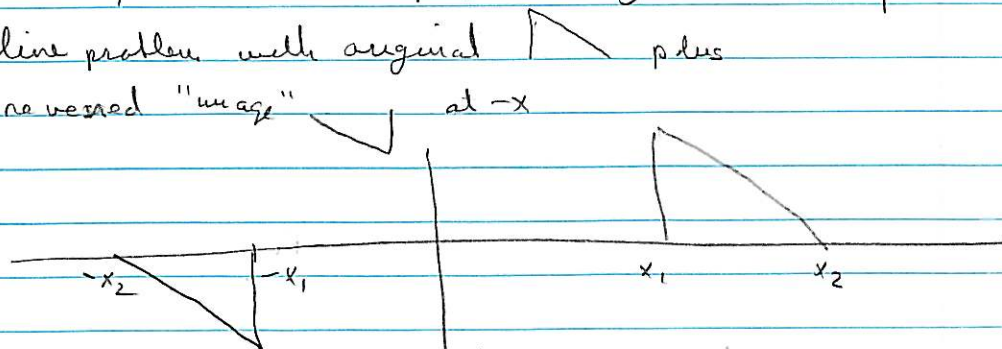
- ii) at $t = t_1 + \epsilon$, the left copy begins process of reflection from body. (hard to draw accurately but it's something like $\rightarrow$)



- iii) For $t > t_2 = x_2/c$ the reflection process is complete, and we are left with two right moving waves.



We get exactly same sol in physical region $x > 0$, if we consider full line problem with original plus -ve reversed "image" at $-x$



with this initial data, the sol of full-line problem is

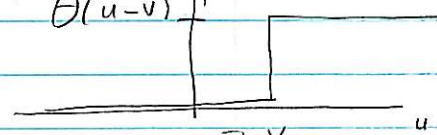
$$u(x, t) = \frac{1}{2} [f(x-ct) + f(x+ct)]$$

We reproduce previous picture of $1/2$ -line sol., if we simply restrict attention to region $x > 0$ in the formula. The purpose of the "image" pulse is to exactly cancel the $1/2$ strength $\swarrow$

at the bdy at $x=0$. Note: it was reasonably easy to visualize situation for initial data ∇ of compact support, but it is valid for any initial data on $(0, \infty)$ with $f(v) \rightarrow 0$ as $x \rightarrow \infty$.

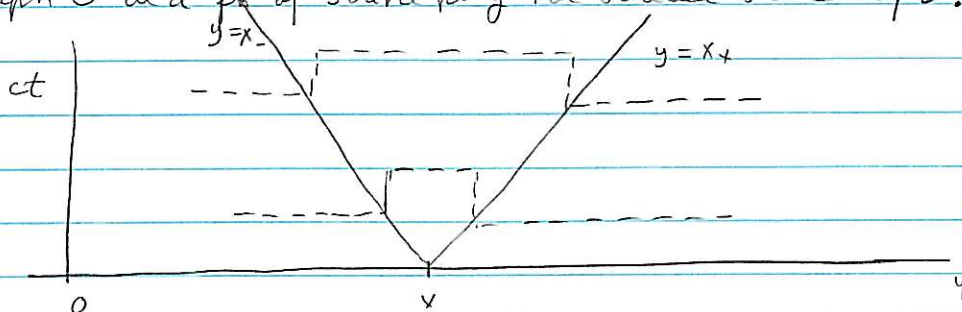
Rewrite d'Alembert formula in Green's fn form: we will simply postulate the Green's fn and show that it works. Later we will derive G from a more fundamental viewpoint.

Our G will be constructed from step fn: $\theta(u-v)$ (called $H(u-v)$ in Hataremann)



Define: $G(x-y, t) = \frac{1}{2c} [\theta(y-x+ct) - \theta(y-x-ct)]$

Graph G as a fn of source pt y for various values of t . Use $x_{\pm} = x \pm ct$



In words: $G(x-y, t) = \begin{cases} 0 & \text{if } y \text{ is outside fwd. light cone from } (x, 0) \\ \frac{1}{2c} & \text{if } y \text{ is inside " " " " " "} \end{cases}$

Clearly the v -part of d'A. sol. can be written as

$$v(x, t) = \int_{-\infty}^{\infty} dy G(x-y, t) g(y)$$

Notice that $\frac{\partial}{\partial t} G(x-y, t) = \frac{1}{2} [\delta(y-x+ct) + \delta(y-x-ct)]$

Using then the full d'A sol can be written as

$$u(x, t) = \int_{-\infty}^{\infty} dy \left[G(x-y, t) g(y) + \frac{\partial}{\partial t} G(x-y, t) f(y) \right] \quad *$$

Since we already know the sol., why is it useful to rewrite it with Green's fn?

i) We can use G to solve wave eqn with source:

$$\left(\frac{1}{c^2} \partial_t^2 - \partial_x^2\right) u = g(x, t)$$

ii) "structure" of soln can be compared with $d \geq 2$, which we will study soon

iii) We can apply image charge ideas to G and use it to write sol of $1/2$ line problems, even including non-trivial data

Ex: Properties of G : i) sol of $\left(\frac{1}{c^2} \partial_t^2 - \partial_x^2\right) G = 0$

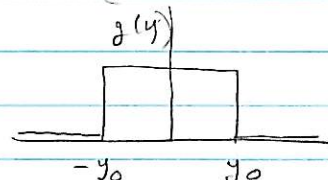
because arguments of step fn are $x_{\pm} \equiv x \pm ct$.

ii) IC's $G(x-y, 0) = 0$ $\partial_t G(x-y, 0) = \delta(x-y)$
trivial poso data $\uparrow$ pts. source of velocity data $\uparrow$

Similar to IC's for $G(t, s)$ for Sturm-Liouville Eqns

iii) symmetry $G(x-y, t) = G(y-x, t)$ $\leftarrow$ (not so obvious)

Ex: Sketch future evolution of initial velocity data of compact support



Ex: The Green's fn for the Dirichlet $1/2$ -line problem is

$$G_D(x, y, t) = \underset{\text{direct term}}{G(x-y, t)} - \underset{\text{image term}}{G(x+y, t)}$$

Verify BC $G_D(0, y, t) = 0$ and sym $G_D(x, y, t) = G_D(y, x, t)$

The sol of IVP on $1/2$ -line is:

$$u(x, t) = \int_0^\infty dy \left\{ G_D(x, y, t) f(y) + \partial_t G_D(x, y, t) g(y) \right\}$$

Same "structure" as $1/2$ for full line.

Wave eqn on drumhead

$$\mathcal{R} \subset \mathbb{C} = \partial\mathbb{R}$$

Think of $\mathcal{R}$

as prototype for wave eqn in finite V in any dim:

$$\text{PDE } \frac{1}{c^2} \partial_t^2 u(\vec{x}, t) = \nabla^2 u(\vec{x}, t) \quad x \in \mathbb{R}$$

$$\text{BC } u(\vec{x}, t) \Big|_{\vec{x} \in \mathcal{C}} = 0 \quad \text{IC } u(x, 0) = f(\vec{x}) \quad \partial_t u(x, 0) = g(\vec{x})$$

Program: Apply traditional methods of sep of var, etc to derive sol of IVP in Gm. fn form. Deduce universal spectral rep of Green's fn for wave eqn.

1) Look for product sols

$$u(\vec{x}, t) = T(t) \phi(\vec{x}) \quad \text{which satisfy PDE + BC's}$$

Usual subset of separable form in PDE

$$\frac{1}{c^2 T(t)} \ddot{T}(t) = \frac{1}{\phi(\vec{x})} \nabla^2 \phi(\vec{x}) = -\lambda$$

So the spatial modes are our old friends the Dirichlet eigenfns: $\{\phi_n(\vec{x})\}$

$$\nabla^2 \phi_n = -\lambda_n \phi_n \quad \phi_n \Big|_{\mathcal{C}} = 0$$

We know that $\lambda_n > 0$ and $\mathcal{C}$ the set is complete

The T problem is $\ddot{T}_n = -\lambda_n c^2 T_n(t)$, general sol is

$$T_n(t) = A_n \cos(\omega_n c t) + B_n \sin(\omega_n c t) \quad \omega_n = \sqrt{\lambda_n}.$$

↪ These describe oscillations in t with angular freq $\omega_n c$ and freq $\nu_n = \omega_n c / 2\pi$

As we have done in the past, we consider superpos.

$$u(\vec{x}, t) = \sum_{n=1}^{\infty} [A_n \cos(\omega_n c t) \phi_n(\vec{x}) + B_n \sin(\omega_n c t) \phi_n(\vec{x})]$$

↪ This satisfies PDE + BC's. We now show how to

find A_n, B_n from IC's.

1) at $t=0$ we impose IC $u(\vec{x}, 0) = f(\vec{x}) = \sum_{n=1}^{\infty} A_n \phi_n(\vec{x})$, so the A_n are the usual coeff in the eigenfn expansion of the position data: $A_n = \frac{1}{N_n^2} (\phi_n, f) = \frac{1}{N_n^2} \iint_{\mathcal{R}} d^2y \phi_n(\vec{y}) f(\vec{y})$

2) to compute B_n we take

$$\partial_t u(\vec{x}, 0) = g(\vec{x}) = c \sum \omega_n B_n \phi_n(\vec{x})$$

$$\text{So } B_n = \frac{1}{c \omega_n N_n^2} (\phi_n, g) \quad \left\{ \begin{array}{l} f(\vec{x}) = a \phi_{n_0}(\vec{x}) \\ g(\vec{x}) = 0 \end{array} \right.$$

Physical interp of solution: Take very special IC

the initial displacement of the drumhead is the shape of a single eigenfn $\phi_{n_0}(\vec{x})$

$$\text{Then } A_n = \begin{cases} a & n = n_0 \\ 0 & n \neq n_0 \end{cases} \quad B_n = 0$$

The future evolution is then $u(\vec{x}, t) = a \cos(\omega_{n_0} c t) \phi_{n_0}(\vec{x})$.

continuous oscillation at freq. $\omega_{n_0} c$ and in the shape of a single eigenfn

The general motion is a complex superp. of many normal modes. For circular drumhead the normal modes involve Bessel fns. See Sec 7.7 and Fig 7.8.3 in H_0 .

Find Green's fn by (a variant of) usual method of subit of A_n, B_n and exchange $\sum_n \leftrightarrow \iint d^2y$.

Our sol $\times$ for 1-dim case suggests that $G(\vec{x}, \vec{y}, t)$ should be found from the velocity data part of sol, so we write

$$\begin{aligned}
 v(\vec{x}, t) &= \sum_{n=1}^{\infty} \sin(\omega_n t) \phi_n(\vec{x}) B_n \\
 &= \iint_R d^2y \underbrace{\sum_{n=1}^{\infty} \frac{\sin(\omega_n t)}{\omega_n c N_n^2} \phi_n(\vec{x}) \phi_n(\vec{y})}_{G(\vec{x}, \vec{y}, t)} g(\vec{y})
 \end{aligned}$$

this defines Grn. fu.
for wave eq. in its spectral form.

Then since $\frac{\partial}{\partial t} \sin(\omega_n t) = \omega_n c \cos(\omega_n t)$,
the full set of IVP can be written as

$$u(\vec{x}, t) = \iint_R d^2y \left[\partial_t G(\vec{x}, \vec{y}, t) f(\vec{y}) + G(\vec{x}, \vec{y}, t) g(\vec{y}) \right]$$

Comments: i) we have found general sol of IVP in a quite straightfwd way using the usual 18.303 tools of sep of vars + eigenfun expts.
 ii. Of course it is only a 'formal' soltn, since we only know G in its spectral form

$$G(\vec{x}, \vec{y}, t) = \sum_{n=1}^{\infty} \frac{\sin(\omega_n t)}{\omega_n c N_n^2} \phi_n(\vec{x}) \phi_n(\vec{y}) g(\vec{y})$$