Problem Set # 08, 18.300 MIT (Spring 2022)

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Due: Last day of lectures (turn it in via the canvas course website).

Contents

1 Introduction. Experimenting with Numerical Schemes

Consider the numerical schemes that follow after this introduction, for the specified equations. **YOUR TASK here is to** *experiment (numerically) with them so as to answer the question:* **Are they sensible?** Specifically:

i.1 Which schemes give rise to the type of behavior illustrated by the "bad" scheme in the GBNS_lecture script?[1] of the 18.311 MatLab Toolkit?

i.2 Which ones behave properly as Δx and Δt vanish?

Further: **show that** they all arise from some approximation of the derivatives in the equations, similar to the approximations used to derive the "good" and "bad" schemes used in the GBNS_lecture script[2] of the 18.311 MatLab Toolkit. That is: **show that the schemes are consistent.**‡

‡ I **strongly recommend** that you read the *Stability of Numerical Schemes for PDE's* notes in the course WEB page **before** you do these problems.

Remark 1.1 *Some of the schemes are "good" and some are not. For the "good" schemes restrictions are needed (as Δx gets small) on Δt to avoid bad behavior — i.e.: fast growth of grid-scale oscillations. Specifically:* **in all the schemes a parameter appears: $\boldsymbol{\lambda = \Delta t/\Delta x}$** *in some cases, and* $\boldsymbol{\nu = \Delta t/(\Delta x)^2}$ *in others. You* **will need to keep this parameter smaller than some constant** *to get the "good" schemes to behave. That is:* $\boldsymbol{\lambda < \lambda_c}$*, or* $\boldsymbol{\nu < \nu_c}$. **For the "bad" schemes, it will not matter how small λ (or ν) is. Figuring out the values of these constants is also part of the problem.** *For the assigned schemes the constants, when they exist ("good" schemes), are* **simple $O(1)$ numbers**, *somewhere between 1/4 and 2. That is, stuff like 1/2, 2/3, 1, 3/2, etc. —* **not** *things like $\lambda_c = \pi/4$ or $\nu_c = \sqrt{e}$.* **You should be able to find them** *by careful numerical experimentation.* ♣

Remark 1.2 *In order to do these problems you may need to write your own programs. If you choose to use MatLab for this purpose, there are several scripts in the 18.311 MatLab Toolkit that can easily be adapted for this. The relevant scripts are:*

- *The schemes used by the* GBNS_lecture *MatLab script are implemented in the script* InitGBNS.
- *The two script series*

  PS311_Scheme_A, PS311_Scheme_B, ... *and* PS311_SchemeDIC_A, PS311_SchemeDIC_B, ...,

  have several examples of schemes already setup in an easy to use format. Most of the algorithms here are already implemented in these scripts. The scripts are written so that modifying them to use with a different scheme involves editing only a few (clearly indicated) lines of the code.

[1] Alternatively: check the *Stability of Numerical Schemes for PDE's* notes in the course WEB page.

[2] Alternatively: check the *Stability of Numerical Schemes for PDE's* notes in the course WEB page.

- *Note that the scripts in the 18.311 MatLab Toolkit are written avoiding the use of "for" loops and making use of the vector forms MatLab allows — things run a lot faster this way. Do your programs this way too, it is good practice.* ♣

Remark 1.3 **Do not** *include lots of graphs and numerical output in your answers. Explain what you did and how you arrived at your conclusions, and illustrate your points with a few selected graphs.* ♣

Remark 1.4 **In all cases the notation:** $x_n = x_0 + n\Delta x$, $t_k = t_0 + k\Delta t$, **and** $u_n^k = u(x_n, t_k)$, **is used.** ♣

2 GBNS02. scheme B. Backward differences for $u_t + u_x = 0$

Statement: Scheme B. Backward differences for $u_t + u_x = 0$

Equation: $u_t + u_x = 0$. Scheme: $u_n^{k+1} = u_n^k - \lambda\,(u_n^k - u_{n-1}^k)$, where $\lambda = \dfrac{\Delta t}{\Delta x}$.

Reminder/read the introduction! Here you are asked to (numerically) study this scheme and decide if it is unstable or not. Specifically, for this scheme:

Is there some positive value of λ, $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, the solution of the scheme converges to the solution to the PDE as $\Delta x \to 0$? In this case **the scheme is stable.**

Else, for any $\lambda > 0$, as $\Delta x \to 0$, the solutions develop very large amplitude grid scale oscillations. Then **the scheme is unstable.**

If there is a λ_c, find it (approximately) by experimenting (numerically) with the scheme.

WARNING: even a stable scheme blows up if $\lambda > \lambda_c$. Observing blow up for one λ is not enough to conclude instability. This does not mean you have to check arbitrarily small λ's — typically $\lambda_c = O(1)$. Note also that, to observe the blow up you need to run many time steps. As $\Delta x \to 0$, keep the time interval over which you solve the equation fixed, say: from $t = 0$ to $t = 1$.

3 GBNS03 scheme C. Centered differences for $u_t + u_x = 0$

Statement: Scheme C. Centered differences for $u_t + u_x = 0$

Equation: $u_t + u_x = 0$. Scheme: $u_n^{k+1} = u_n^k - \dfrac{1}{2}\lambda\,(u_{n+1}^k - u_{n-1}^k)$, where $\lambda = \dfrac{\Delta t}{\Delta x}$.

Reminder/read the introduction! Here you are asked to (numerically) study this scheme and decide if it is unstable or not. Specifically, for this scheme:

Is there some positive value of λ, $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, the solution of the scheme converges to the solution to the PDE as $\Delta x \to 0$? In this case **the scheme is stable.**

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If there is a λ_c, find it (approximately) by experimenting (numerically) with the scheme.

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4 Experiments with a slinky

Statement: Experiments with a slinky

Consider a homogeneous cylindrical rod (made of an elastic material), subject to (small amplitude) longitudinal deformations. Let $\boldsymbol{x}$ be the length coordinate (measured along the axis of the cylinder) when the cylinder is in its relaxed position. Use x as a label for the mass elements in the cylinder.[3] For every mass element x, let $\boldsymbol{u = u(x,\, t)}$ be its position at time $\boldsymbol{t}$, measured along the axis of the cylinder (note that $\boldsymbol{u = x}$ corresponds to the cylinder at rest.) Then u describes the state of the cylinder at any time and obeys the **wave equation:**

$$u_{tt} - c^2\, u_{xx} = 0, \quad \text{where} \quad c = \sqrt{k/\rho}. \tag{4.1}$$

Here $\boldsymbol{\rho}$ is the density (mass per unit length) of the rod, and $\boldsymbol{k}$ characterizes the elastic response of the material: if we stretch the cylinder by an amount ΔL, then the elastic force is $k\dfrac{\Delta L}{L}$, where L is the length of the rod (note that $\boldsymbol{k}$ has the dimensions of a force, thus c is a speed).

Remark 4.1 *The basic assumption here is that the cylinder remains at all times within the regime of applicability of Hooke's law. This means that the deformations (given by $u_x - 1$) are small enough everywhere. In particular, this also implies that variations in the cross-section of of the cylinder can be ignored (e.g.: if volume is preserved, the cross section will be larger in regions under compression than in those under tension).*

In the derivation of equation (4.1) it is assumed that the elastic forces are dominant, so that other forces (e.g.: gravity) can be ignored. For a rod with a vertical orientation, such that the elastic forces are not dominant over gravity, equation (4.1) must be modified to:

$$u_{tt} - c^2\, u_{xx} = -g, \tag{4.2}$$

where $\boldsymbol{g}$ is the acceleration of gravity, and we assume that the vertical coordinate $\boldsymbol{x}$ increases upwards. In particular, **consider the case of a rod hanging vertically without any motion** (i.e.: $u = u(x)$, with no time dependence), and measure $\boldsymbol{x}$ from the bottom of the rod. Then:

$$u = 0 \quad \text{and} \quad u_x = 1 \ \text{ at } \ x = 0, \tag{4.3}$$

where the second equation follows because there is no force at the lower end (no section of the rod below that must be supported). Then the equation for $u = u(x)$, namely:

$$c^2\, u_{x\,x} = g,$$

can be integrated to yield:

$$\boldsymbol{u = \frac{g}{2c^2}\, x^2 + x.} \tag{4.4}$$

A particular **example where this should apply to is that of a slinky. One objective of this problem is for you to check how well a slinky obeys equation (4.4).**

Proceed as follows:

1. Get a slinky in good condition and draw a straight line along its edge, parallel to the slinky's axis. Draw the line so that, when it reaches one end (the "bottom" end), it does so **at the end** of the coil that makes the slinky — i.e.: no more coil beyond the mark.
2. Starting from the "bottom" end of the slinky, name the points at which each coil is marked by the line as $n = 0, 1, \ldots$ Then (if $\boldsymbol{w}$ is the width of a coil) when the slinky is at rest, the position of the $\boldsymbol{n^{th}}$ point is given by
$$x_n = n\, w.$$
3. To find w, measure the total length of the slinky, and divide this by the number of coils. You can also easily measure the "density" ρ of the coil by weighting it and dividing the result by its length.
4. Hang the slinky in a vertical position[4] (with the bottom end down) and wait till it is at rest. Then measure

[3] Since we are considering only longitudinal motions, points in a cross section move in unison and we need not label them separately.
[4] For example, staple it to the underside of a shelf by a wall.

the distance u_n of the n^{th} point from the point $n = 0$ at the bottom. One way to do this is to have a measuring tape on a wall right behind the hanging slinky.

5. Equation (4.4) predicts that
$$u_n = \frac{g}{2c^2}\, x_n^2 + x_n = \frac{g\rho}{2k}\, x_n^2 + x_n. \tag{4.5}$$

6. **The question is now:** How well does equation (4.5) match your measurements? Of course, you do not have k, but you will have several values of n. If (4.5) applies, then
$$u_{n+1} - u_n = \Delta u_n = \frac{g\,\rho}{k}\, w^2\, n + \frac{g\,\rho}{2\,k}\, w^2 + w.$$
Thus a plot of $\boldsymbol{u_{n+1} - u_n}$ versus $\boldsymbol{n}$ should give a straight line with slope $\boldsymbol{g\,\rho\, w^2/k}$. From this you can get $\boldsymbol{k}$, which is the hardest quantity to measure directly in this context.

Next suspend the slinky from one end and set it to vibrate (longitudinally). In this case, if we set the origin for the coordinate x at the top (where the slinky does not move), the governing equation will still be (4.2), but the boundary conditions are now:
$$u(0,\, t) = 0 \quad \text{and} \quad u_x(-L,\, t) = 1, \tag{4.6}$$
where **$\boldsymbol{L}$ is the length of the slinky in its rest state.** The first condition here follows because $x = 0$ corresponds to the clamped end at the top, while the second simply states that there are no elastic forces at the bottom end (same reason used when deriving the second condition in (4.3)). It is easy to see that these conditions (and the equation) are satisfied by the function
$$u = a \sin\left(\frac{\pi}{2\,L}\, x\right) \sin\left(\frac{c\,\pi}{2\,L}\, t\right) + \frac{g}{2\,c^2}\, x^2 + \left(1 + \frac{g\,L}{c^2}\right)\, x, \tag{4.7}$$
where a is an arbitrary constant. This solution corresponds to an oscillation with **period $\boldsymbol{T = \frac{4\,L}{c}}$.**

Now, **continue the experiment:**

7. In the situation described in the paragraph above, measure the period of the slinky — do not try to measure a single period, time several and then divide by the number of periods timed.

8. Compare the result of your measurement of the period with the one given by the formula for T above — from the prior steps you can obtain a value for c.

9. Discuss the results of your experiment.

THE END.