

Problem Set # 06, 18.300 MIT (Spring 2022)

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Due: Fri April 22 (turn it in via the canvas course website).

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The "computer exercise in Fourier Series" is part of the NEXT problem set. It is included here so you can start early with it. Include your answer with p-set #7, NOT with p-set #6.

1 Computer Exercise in Fourier Series

Statement: Computer Exercise in Fourier Series

1.0.1 Introduction: Fourier Series

Generally, a 2π -periodic function $F = F(x)$ can be expressed in terms of its **Fourier Series** where the n^{th} complex Fourier coefficient F_n is defined by

$$F(x) = \sum_{n=-\infty}^{\infty} F_n e^{i n x}, \quad (1.1)$$

$$F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-i n x} dx, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (1.2)$$

An alternative formulation, obtained upon using $e^{-i n x} = \cos(n x) - i \sin(n x)$, is given by:

$$F(x) = c_0 + \sum_{n=1}^{\infty} (c_n \cos(n x) + s_n \sin(n x)), \quad (1.3)$$

where: (i) $c_0 = F_0$ and $c_n = (F_n + F_{-n})$ are the cosine Fourier coefficients, and (ii) $s_n = i (F_n - F_{-n})$ are the sine Fourier coefficients. Thus

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx, \quad (1.4)$$

$$c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(n x) dx, \quad \text{for: } n = 1, 2, 3, \dots \quad (1.5)$$

$$s_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(n x) dx, \quad \text{for: } n = 1, 2, 3, \dots \quad (1.6)$$

If F is real valued, then F_{-n} is the complex conjugate of F_n , so that

$$c_n = 2 \operatorname{Re}(F_n) \quad \text{and} \quad s_n = -2 \operatorname{Im}(F_n) \quad \text{for } n > 0. \quad (1.7)$$

Generally, the issue of how well (or even in which sense), the Fourier Series in (1.1) or in (1.3) converges to the function F is a rather subtle one. *The main point of this problem is to conduct a numerical exploration of some aspects of this question.* In particular,

consider the **partial sums**:

$$F_N(x) = c_0 + \sum_{n=1}^N (c_n \cos(nx) + s_n \sin(nx)), \quad (1.8)$$

where N is some natural number.

Important questions are then: **How well does F_N approximate the function F ?** and **How big is the error¹ and how fast does it vanish as $N \rightarrow \infty$?**

Remark 1.1 *An important element in answering the questions above is how fast the Fourier coefficients vanish as $n \rightarrow \infty$. This is determined by how fast the*

power spectrum

$$P_n = \frac{1}{2}|F_n| = \sqrt{c_n^2 + s_n^2} \quad (1.9)$$

vanishes as $n \rightarrow \infty$ (assume F is real valued, so that (1.7) applies).

P_n gives information on “how important” the n -th mode is in the Fourier Series. The name follows from the fact that in many physical situations one can interpret the square of the amplitude of the n -th Fourier coefficient, P_n^2 , as the amount of energy in the n -th mode of the solution — e.g.: for the wave equation.

1.0.2 The problem to be done

This problem objective is to “experimentally” **study how Fourier series converge**. For this purpose you should use the following MatLab scripts

FouSerRedame.m fourierSC.m FSFun.m FSoption.m FSoptionP.m heatSln.m MakeButtonFSC.m

Put the scripts in a directory and start MatLab there. The help command will work as usual, in particular: help FouSerReadme gives a description of all the scripts. Each script has its own detailed description. The **script you need is fourierSC**. The others (except for heatSln) are helper scripts.

IMPORTANT:

- When you start the script fourierSC, it will ask you the questions:
 - A.** *Do you want to use the fancy (with buttons) or the plain interface?*
 - B.** *Up to how many terms in the Fourier series do you want to compute?*
 - C.** *For which values you want to plot?*

About **B** and **C**: Calculations will be done (and the results shown) for the partial sums in (1.8), for the values $N = 0, N_{\text{skip}}, 2*N_{\text{skip}}, \dots, N_p$. You will be asked to input N_{skip} and N_p . About **A**: the fancy version does not work for some versions of MatLab.

- After you finish answering the questions, fourierSC will present you with a list of options for functions whose Fourier series it can compute: “user’s choice”, and pre-selected. **Check the scripts code to make sure you that understand exactly which functions you are dealing with.**
- The script FSFun.m is the one used to input the “user’s choice” selection — whatever function you program there will be the one used when “user’s choice” is selected. A trivial example is pre-programmed in FSFun.m, but **you should alter it**, and write there any function for which you want to investigate the Fourier series, **to go beyond** the preselected options, which I encourage you to do.
- The pre-selected options include smooth functions, as well as functions with various types of singular behaviors — discontinuities, corners and cusps. *The idea is to investigate how any particular “singular” behavior in the function is related to the convergence properties of its Fourier series.*

A **cusp is a singularity** such as the one that $\sqrt{|x|}$ has at the origin. Other possibilities are $|x|^\alpha$, where $0 < \alpha < 1$. *Investigate the effect of singularities of this type on the convergence.*

The most important singular behavior whose effect on the Fourier series you should elucidate is that of a discontinuity. How does it affect the convergence? How do the partial sums look like in this case? Is there any peculiar behavior you can observe?

¹ Note also that there are many ways that the error can be measured: point-wise; least-squares; etc.

Odd and even functions are also provided in the pre-selections, so that you can see what effect symmetries of the function have on its Fourier series. Can you think of other symmetries?

- The script `fourierSC` makes lots of plots, which will be made one on top of the other. You *need to move the windows to see all the plots*. **These plots illustrate various aspects of how a Fourier series behaves, as follows** (this is the order in which the plots are done):
 - **Exact function whose Fourier series is being computed.**
 - **Sine Fourier coefficients s_n , as a function of n .**
 - **Cosine Fourier coefficients c_n , as a function of n .**
 - **Semi-log plot of the power spectrum $P_n = \sqrt{c_n^2 + s_n^2}$ as a function of n** — exponential decay yields a straight line in this kind of plot.
 - **Log-log plot of the power spectrum as a function of n** — algebraic decay yields a straight line in this kind of plot.
 - **Partial sums $F_N = F_N(x)$ — as in equation (1.8) — for $N = 0:Nskip:Np$.** All these plots will be shown in the same window, so you must look at them as they are done.
 - **Relative error in the approximation F_N , as a function of N** — shows the error in the partial sums in (1.8), as a function of N , for $N = 1:Np$.
 - **Semi-log plot of the relative error, as a function of N .**
 - **Log-log plot of the relative error, as a function of N .**

This is what you are expected to do: Use the script `fourierSC` and experiment with the various choices. Then report any “pattern” or peculiar behavior you observe in the way Fourier series converge. The plots are useful in figuring out how fast things converge (e.g. how fast do the Fourier coefficients vanish as $n \rightarrow \infty$). Look at the plots, look for patterns and trends. Make hypothesis as to what is happening and **test them by further experimentation** — use the script `FSFun` to produce functions where you can test your hypothesis. Write your conclusions in the answers. **Describe the evidence for your conclusions** — no proof is required, numerical evidence is enough, but **you must produce, and describe, the evidence!** Think of it in the same way that you would think in the situation of a lab experimenter trying to figure out what happens in some problem. **A few plots with your answer are fine, but please, just a few!**

Important: the point of this problem is not for you to run code and show pretty pictures of what you see. I want you to arrive at quantitative conclusions about how Fourier series converge, etc. The plots for the pre-programmed functions will show you patterns that you need to recognize, and test with other functions to identify what causes them. I **expect statements that sound like this:** “if a function has the following property ... (discontinuity, cusp, corner, whatever), then the power spectrum decays like ...; the error in F_N behaves like ...; etc.”

Important: Anything smaller than about 10^{-14} is numerical error. Ignore it!

2 Pendulum with torsion

Statement: Pendulum with torsion

Consider an horizontal axle A , of length ℓ , with a rod R of length L attached (perpendicularly) at its **midpoint**. Let a mass M be attached to the other end of the rod R — see figure 2.1. Assume that:

1. The axle A is free to rotate, and we can ignore any frictional forces (i.e.: they are small).
2. Any deformations to the axle and rod shapes are small enough that we can ignore them. They are both straight cylinders, and they remain so.

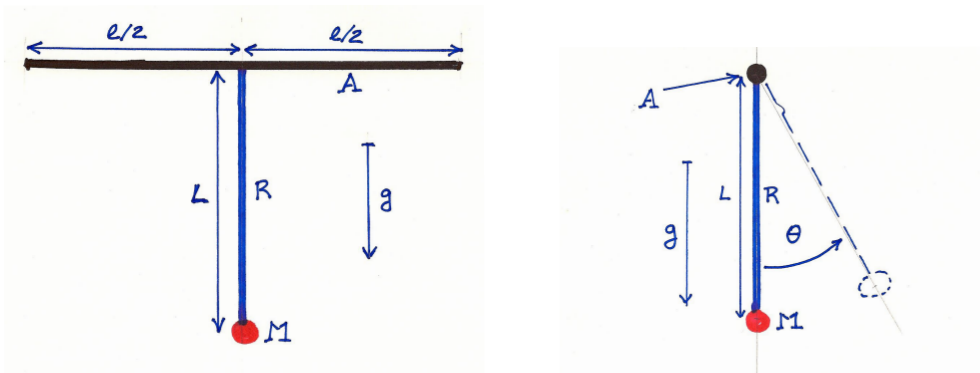


Figure 2.1: Problem: Pendulum with torsion. Left: side view of the “axle and rod with a mass” device in this problem. Right: view along the axle A . The rod is constrained to move in the plane perpendicular to the axle, which can only rotate. Hence the only dynamical variable is the angle θ .

3. The masses of both the rod and the axle are small compared to M , so we ignore them (this assumption is not strictly needed, but we make it to make matters simple).

In this case the mass M is restricted to move on a circle in a vertical plane, perpendicular to the axle. We can describe the state of the system at any time by the angle θ that the rod R makes with a vertical line pointing down. This angle satisfies the **pendulum** equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta, \quad (2.1)$$

where g is the acceleration of gravity and t is time.

Imagine now that the axle A is not free to rotate, but in fact it is fixed at one (or both) ends, so that it opposes the motion of the rod by elastic torsional forces. Assume that, when the rod is pointing down, these forces vanish (i.e.: the axle has no twist). Then, in the case where the angle θ is small enough that the torsional forces are proportional to the angle of twist (Hooke’s law), **derive the equation that θ must satisfy in the two cases: (1) one end of the axle is fixed, and the other is free to rotate (without friction), or (2) both ends are fixed.**

Hint 2.1 *In the Hooke’s law regime, for a given fixed bar, the torque generated is directly proportional to the angle of twist, and inversely proportional to the distance over which the twist occurs.*

To be specific: in the problem here, imagine that a section of length $\Delta\ell$ of the axle has been twisted by an amount (angle) Ψ . Then, if T is the torque generated by this twist, we have

$$T = \frac{\kappa \Psi}{\Delta\ell},$$

where κ is a constant that depends on the axle material and the area of its cross-section (assuming that the axle is an homogeneous cylinder). **What are the dimensions of κ ?**

This torque translates on a tangential force of magnitude $F = T/L$ on the mass M at the end of the rod R of length L . The sign of the force is such that it acts to undo the twist.

Finally, note that in this problem, when the rod is at an angle θ from its vertical reference position, a twist of magnitude θ will be generated on any side where the axle ends are tied.

3 Two torsion coupled pendulums

Statement: Two torsion coupled pendulums

A variation on the problem “*Pendulum with torsion*” is as follows:

1. The axle has two equal rods attached to it (equally spaced along the axle, so that they are a distance $\ell/3$ from each other and from the ends).
2. Each rod has a mass $\frac{1}{2} M$ at its end.
3. The axle is free to rotate without friction.
4. The axle has no twist when both rods are pointing down.

Write the equations of motion for this system. Note that the state of the system is described by two angles, and that when writing the equations of motion you have to consider the torsional forces that are generated when the two angles are not equal. Assume that the angle differences are small, so Hooke’s law applies and the torsional forces are proportional to the angle difference.

4 SGDP04. Simple waves and Riemann invariants

Statement: Simple waves and Riemann invariants

Consider the equations of isentropic Gas Dynamics $\rho_t + (\rho u)_x = 0$ and $u_t + u u_x + \frac{c^2}{\rho} \rho_x = 0$, (4.1)

where $p = p(\rho)$ is increasing and $c = \sqrt{\frac{dp}{d\rho}}$. In the notation of remark 4.1, here

$$\vec{Y} = \begin{pmatrix} \rho \\ u \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}. \quad (4.2)$$

Consider now the simple waves² associated with

$$\lambda = u + c \quad \text{and} \quad \vec{R} = \begin{pmatrix} 1 \\ c/\rho \end{pmatrix}. \quad (4.3)$$

Show that these are exactly the same solutions that result from taking $u - h = \text{constant}$ in (5.20–5.22). See hint 4.2.

Remark 4.1 Simple wave solution. Consider a system of equations

$$\vec{Y}_t + \mathbf{A}(\vec{Y}) \vec{Y}_x = 0, \quad (4.4)$$

where $\vec{Y} = \vec{Y}(x, t)$ is column N -vector valued, and \mathbf{A} is $N \times N$ -matrix valued.

Let $\lambda = \lambda(\vec{Y})$ be a real eigenvalue of \mathbf{A} , with corresponding right-eigenvector \vec{R} ,

$$\mathbf{A} \vec{R} = \lambda \vec{R}. \quad (4.5)$$

Let $\vec{U} = \vec{U}(\phi)$ be a solution of the ode

$$\frac{d\vec{U}}{d\phi} = \vec{R}(\vec{U}). \quad (4.6)$$

Then it can be shown that

$$\vec{Y} = \vec{U}(\phi(x, t)) \quad (4.7)$$

is a solution to (4.4) provided that ϕ solves the scalar equation

$$\phi_t + \lambda(\vec{U}(\phi)) \phi_x = 0. \quad (4.8)$$

Solutions of this type are called “simple waves”, because they “excite”

a single eigendirection of \mathbf{A} . Note that the solutions to (4.8) generally exhibit

wave-steepening (and develop infinite derivatives in finite time) if

$$\mathbf{0} \neq \frac{d\lambda}{d\phi} = (\nabla \lambda) \cdot \vec{R}, \quad (4.9)$$

where we have used the chain rule, and $\nabla \lambda$ denotes the gradient of λ with

respect to \vec{Y} . An eigenvalue that satisfies $(\nabla \lambda) \cdot \vec{R} \neq 0$ is called **genuinely nonlinear**,

while one satisfying $(\nabla \lambda) \cdot \vec{R} \equiv 0$ is said to be **linearly degenerate**. ♣

Hint 4.2 Note: $u - h = \text{constant}$ trivially satisfies (5.22). Further, since the definition of h in (5.20) involves an arbitrary constant of integration, we can assume (without loss of generality) $u - h = 0$. That is $u = h(\rho)$. Then (5.21) becomes

$$(u + h)_t + (u + c)(u + h)_x = 0. \quad (4.10)$$

² Note that (4.6) has more than one solution.

But $u + h = 2h$. Hence, using the chain rule and dividing by $2 dh/d\rho = 2c/\rho \neq 0$, (5.21) yields
where $u + c = h(\rho) + c(\rho)$.

$$\rho_t + (u + c) \rho_x = 0, \quad (4.11)$$

♣

5 1-D isentropic Euler equations of Gas Dynamics (supplementary)

Equations that govern the behavior of a gas can be derived using the same *Conservation Equation* techniques used to derive equations for the examples of Traffic Flow, River Flows, Shallow Water Waves, Modulations of Dispersive Waves, etc. The conserved quantities in this case are the mass, the momentum and the energy. The resulting equations are the **Euler equations of Gas Dynamics**.

Under certain conditions, one can assume that the entropy is a constant throughout the flow. Then the equations can be simplified, with the elimination of the equation for the conservation of energy. The **one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics** are

$$\left. \begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \end{aligned} \right\} \quad (5.12)$$

where $\rho = \rho(x, t)$, $u = u(x, t)$, and $p = p(x, t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation here implements the conservation of mass, and the second the conservation of momentum. We also need an equation relating the fluxes of the conserved quantities with the conserved densities — the analogue of the equation $q = Q(\rho)$ in Traffic Flow. In this case this is provided by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \quad (5.13)$$

For example, for an ideal gas $P = \kappa \rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

The system of equations given by (5.12) is known as the

Conservation Form of the Isentropic Equations of Gas Dynamics, in Eulerian Coordinates.

Remark 5.2 Normally, the pressure is a function of both the density and some other thermodynamic variable, such as the temperature. But the isentropic assumption allows us to write the pressure as a function of the gas mass density only. ♣

Introduce now the function $c = c(x, t) > 0$ (**c is the sound speed**) by³

$$c = C(\rho), \quad \text{where } C(\rho) = \sqrt{\frac{dP}{d\rho}(\rho)}. \quad (5.14)$$

An alternative form of the equations in (5.12) is then given by

$$\left. \begin{aligned} \rho_t + u \rho_x + \rho u_x &= 0, \\ u_t + \frac{c^2}{\rho} \rho_x + u u_x &= 0. \end{aligned} \right\} \quad (5.15)$$

³ For an ideal gas, $c = \sqrt{\gamma p/\rho}$. For dry air at one atmosphere and 15 degrees Celsius: $p = 1.013 \times 10^6$ dyn/cm², $\rho = 1.226 \times 10^{-3}$ g/cm³, and $\gamma = 1.401$. Hence $c = 340.2$ m/s — measured value is $c = 340.6$ m/s.

Multiplying the first equation here by c/ρ , and adding (or subtracting) it to the second, yields the **characteristic form** of the equations

$$0 = (u_t + (u + c)u_x) + \frac{c}{\rho}(\rho_t + (u + c)\rho_x), \quad (5.16)$$

$$0 = (u_t + (u - c)u_x) - \frac{c}{\rho}(\rho_t + (u - c)\rho_x). \quad (5.17)$$

Equivalently

$$0 = \frac{du}{dt} + \frac{c}{\rho} \frac{d\rho}{dt} \quad \text{along} \quad \frac{dx}{dt} = u + c, \quad (5.18)$$

$$0 = \frac{du}{dt} - \frac{c}{\rho} \frac{d\rho}{dt} \quad \text{along} \quad \frac{dx}{dt} = u - c. \quad (5.19)$$

These last two equations show that, if we **introduce a new variable** $h = h(\rho)$, defined by

$$\frac{dh}{d\rho} = \frac{C(\rho)}{\rho}, \quad (5.20)$$

then

$$u + h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u + c, \quad (5.21)$$

$$u - h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u - c. \quad (5.22)$$

The variable $\mathcal{R} = u + h$ is the **right Riemann invariant**, while $\mathcal{L} = u - h$ is the **left Riemann invariant**.

THE END.