Problem Set # 04, 18.300 MIT (Spring 2022)

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Due: Fri March 18 (turn it in via the canvas course website).

Contents

1 Haberman 7401. Solve initial value problem

Statement: Solve initial value problem

Assume that $u(\rho) = u_m\,(1 - \rho/\rho_j)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x < 0, \\ \rho_0\,(L - x)/L & \text{for } \; 0 \le x \le L, \\ 0 & \text{for } \; L < x, \end{cases} \tag{1.1}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, determine and sketch $\rho(x,\,t)$.

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2 Haberman 7402. Solve initial value problem

Statement: Solve initial value problem

Assume that $u(\rho) = u_m\,(1 - \rho^2/\rho_j^2)$, where u_m is the speed limit and ρ_j is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x < 0, \\ \rho_0\,(L - x)/L & \text{for } \; 0 < x < L, \\ 0 & \text{for } \; L < x, \end{cases} \tag{2.1}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, **determine and sketch** $\rho(x,\,t)$**.**

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3 Haberman 7701. Shock velocity when $u = u(\rho)$ is linear

Statement: Shock velocity when $u = u(\rho)$ is linear

If $u = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$, **what is the velocity of a traffic shock separating the densities ρ_0 and ρ_1?** Simplify the expression as much as possible. Show that the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1.

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4 Haberman 7902. Shock velocity

Statement: Shock velocity

Suppose that

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x > 0, \\ 0 & \text{for } \; x < 0. \end{cases} \tag{4.1}$$

Determine the velocity of the shock. Briefly give a physical explanation of the result. What does this shock correspond to?

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5 KiNe03. Initial value problem with Q quadratic

Statement: Initial value problem with Q quadratic

Consider the traffic flow equation

$$\rho_t + q_x = 0, \tag{5.1}$$

for a flow $q = Q(\rho)$ that is a quadratic function of ρ. In this case $c = \mathrm{d}Q/\mathrm{d}\rho$ is a conserved quantity as well (why?). Thus the problem (including shocks, if any) can be entirely formulated in terms of c, which satisfies

$$c_t + \left(\frac{1}{2}\,c^2\right)_x = 0. \tag{5.2}$$

1. Consider the initial value problem determined by (5.2) and [1]

$$c(x,\,0) = 0 \;\text{ for }\; x \le 0 \quad \text{and} \quad c(x,\,0) = 2\,\sqrt{x} \ge 0 \;\text{ for }\; x \ge 0. \tag{5.3}$$

Without actually solving the problem, **argue that the solution to this problem must have the form**

$$c = t\,f(x/t^2) \;\text{ for }\; t > 0, \quad \text{for some function } f. \tag{5.4}$$

Hint. Let $c = c(x,\,t)$ be the solution. For any constant $a > 0$, define $\mathcal{C} = \mathcal{C}(x,\,t)$ by $\boldsymbol{\mathcal{C} = \frac{1}{a}\,c(a^2\,x,\;a\,t)}$. What problem does $\mathcal{C}$ satisfy? Use now the fact that the solution to (5.2–5.3) is unique to show that (5.4) must apply, by selecting the constant $\boldsymbol{a}$ appropriately at any fixed time $t > 0$. ♣

2. Use the method of characteristics to **solve the problem in (5.2–5.3).** Write the solution **explicitly** for all $\boldsymbol{t > 0}$, and **verify that it satisfies (5.4).** *Warning: the solution involves a square root. Be careful to select the correct sign, and to* **justify your choice.**

3. For the solution obtained in item **2**, **evaluate $\boldsymbol{c_x}$ at $\boldsymbol{x = 0}$ for $\boldsymbol{t > 0}$.** Note that this derivative is discontinuous there, so it has two values (left and right).

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[1] In a traffic problem, c must satisfy $c(\rho_j) \le c \le c(0)$. Ignore the fact that this does not apply for (5.2).

6 Linear 1st order PDE (problem 09)

Statement: Linear 1st order PDE (problem 09)

Surface Evolution. The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its "height" $h = h(x,\, y,\, t)$ relative to the $(x,\, y)$-plane of reference. Under appropriate conditions, a rather complicated pde can be written[2] for h. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r}\, h_r, \quad \text{for } r = \sqrt{x^2+y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \tag{6.1}$$

Axial symmetry is assumed, so that $h = h(r,\, t)$. Obviously, ***h* should be an even function of *r***. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r,\, 0) = H(r^2), \tag{6.2}$$

where H is a smooth function describing a localized bump. Specifically: **(i)** $H(0) > 0$, **(ii)** H is monotone decreasing. **(iii)** $H \to 0$ as $r \to \infty$. **Note that $h(r,\, 0)$** *is an even function of* r.

1. Using the theory of characteristics, write an explicit formula for the solution of (6.1 – 6.2).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2\,A$?
4. Show that the resulting solution is an even function of r for all times.
5. Show that, as $t \to \infty$, the bump shrinks and vanishes. **Hint.** Pick some example function H with the properties above, and plot the solution for various times. This will help you figure out why the bump shrinks and vanishes.

THE END.

[2] From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.