Problem Set # 03, 18.300 MIT (Spring 2022)

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Due: Wed March 9 (turn it in via the canvas course website).

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1 Haberman 6307. Design flow density curve fitting Lincoln Tunnel data

Statement: Design flow density curve fitting Lincoln Tunnel data

Consider exercise 6103.† Suppose that the drivers accelerate in such a fashion that

$$\alpha = u_t + u\,u_x = -\frac{a^2}{\rho}\rho_x, \quad \text{where } a > 0 \text{ is a constant.} \tag{1.1}$$

(a) Physically interpret this situation.

(b) If u only depends on ρ, and the equation for conservation of cars is valid, show that

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = -\frac{a}{\rho}. \tag{1.2}$$

(c) Solve the differential equation in part **(b)**, subject to the condition that $u(\rho_{\text{max}}) = 0$. The resulting flow–density curve fits quite well to the Lincoln Tunnel data.

(d) Show that a is the velocity that corresponds to the road's capacity.

(e) Discuss objections to the theory for small densities.

† Note: exercise 6103 asks to show that the car's acceleration is given by $\alpha = u_t + u\,u_x$. Here just use this.
Another useful result is that from exercise 6306, where the task is to show that . . . . $\alpha = -\rho\,\frac{\mathrm{d}u}{\mathrm{d}\rho}\,u_x$.
This follows by: Substitute $u = u(\rho)$ into $\alpha = u_t + u\,u_x$. Then (chain rule) $\alpha = \frac{\mathrm{d}u}{\mathrm{d}\rho}\,(\rho_t + u\,\rho_x) = \frac{\mathrm{d}u}{\mathrm{d}\rho}\,(u - c)\,\rho_x$ (use the equation in the last step). But $q = u\,\rho$, so that $c = u + \frac{\mathrm{d}u}{\mathrm{d}\rho}\,\rho$. Hence $\alpha = -\rho\,(\frac{\mathrm{d}u}{\mathrm{d}\rho})^2\,\rho_x$. ♣

2 Problem 7104a. PDE satisfied by the wave velocity

Statement: PDE satisfied by the wave velocity

Consider the pde $\rho_t + q_x = \rho_t + c(\rho)\,\rho_x = 0$, where $q = q(\rho)$ and $c = c(\rho) = \dfrac{\mathrm{d}q}{\mathrm{d}\rho}(\rho)$. Assume that $\rho = \rho(x,\,t)$ is a smooth solution of the equation, and let $c = c(x,\,t) = c(\rho)$. **Find the pde that c satisfies**. Note: *the pde is the same no matter what solution ρ is selected! In this sense it is unique, hence the use of "Find the pde", not "Find a pde".*

3 Linear 1st order PDE (problem 03)

Statement: Linear 1st order PDE (problem 03)

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{3.1}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to find the solution. Write, **explicitly**, $\boldsymbol{u = u(x,\,y)}$ as a function of x and y, using f.

Hint. Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$ (where $-\infty < \tau < \infty$). Finally: eliminate τ, to get u as a function of x and y.

Part 2. In which part of the $(x,\,y)$ plane is the solution determined?

Hint. Draw in the x-y plane the characteristics computed in part 1.

Part 3. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

4 TFPb09. Quasi-linear equation solution

Statement: Quasi-linear equation solution

Find the solution to

$$x^2\,\psi_x - \psi^2\,\psi_y = 0, \tag{4.1}$$

with $\boldsymbol{\psi = x}$ on $\boldsymbol{y = x}$, for $\boldsymbol{x > 0}$. **Where is the solution defined, and why?**

5 TFPb10. Solve linear scalar pde using characteristics

Statement: Solve linear scalar pde using characteristics

Find the solution to the equation

$$x\,\phi_y + \phi_x = \phi, \tag{5.1}$$

with $\phi = y$ for $x = 0$ and $y > 0$.

Describe and plot the region where the characteristics reach (thus where the solution is defined).

6 TFPb11. Check solution by implicit differentiation

Statement: Check solution by implicit differentiation

Consider the equation
$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = -\rho. \tag{6.1}$$

A. Show by direct substitution that $\rho = \rho(x, t)$, as defined by the implicit equation

$$\rho = \exp(-t)F(x + \rho) \quad \text{(here } F = F(x) \text{ is an arbitrary function),} \tag{6.2}$$

solves equation (6.1) above. That is: first use implicit differentiation to calculate $\boldsymbol{\frac{\partial \rho}{\partial t}}$ and $\boldsymbol{\frac{\partial \rho}{\partial x}}$. Then substitute the answers to this calculation in the pde, and show directly that the equation is satisfied.
You may assume that ρ and F have as many derivatives as needed.

B. Find the characteristics for equation (6.1) above. Then show how the solution in (6.2) follows from using them, in the same fashion used in the lectures to show that the solution to the Traffic Flow equations can be written implicitly as $\rho = F(x - c(\rho)t)$. Recall that *the characteristics are special curves along which the pde reduces to an ode*

Remark 6.1 *Equation* (6.1) *cannot be a traffic flow equation, since it does not conserve the number of cars: cars are being removed from the road at a rate proportional to the local car density. Well, this could perhaps be explained by a road with lots of big potholes, but there is a bigger problem: the wave speed $c = \rho$ increases with the density ρ, while for traffic flow it should decrease.* ♣

THE END.