Problem Set # 02, 18.300 MIT (Spring 2022)

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Due: Monday February 28 (turn it in via the canvas course website).

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1 Cola07. Small vibrations of a string under tension in 2D

Statement: Small vibrations of a string under tension in 2D

Here you are asked to derive the *equation for the transversal vibrations of a thin elastic string under tension*, under the following hypotheses

1. The string is homogeneous, with a *mass density* (mass per unit length) ρ — ρ is a constant.
2. The motion is restricted to the x-y plane. At equilibrium the string is described by $y = 0$ and $0 \leq x \leq L =$ string length. The *tension* $T > 0$ is then *constant*. See remark **1.1**, #1–3.
3. The string has no bending strength.
4. The amplitude of the vibrations is very small compared with their wavelength, and any longitudinal motion can be neglected. Thus:
   — The string can be described in terms of a *deformation function* $u = u(x,\, t)$,
   such that the equation for the *curve describing the string* is[1] $y = u(x,\, t)$.
   — The tension remains constant throughout — see remark **1.1**, #4.

Use the conservation of the transversal momentum to derive an equation for u. Thus:

(i) Obtain a formula for the transversal momentum density along the string (momentum per unit length).

(ii) Obtain a formula for the transversal momentum flux along the string (momentum per unit time).

(iii) Use the differential form of the conservation of transversal momentum to write a pde for u.

Hint: when doing (ii) remember that forces are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces. Remember also that u_x is very small. Be careful with the signs here! ♣

The pde for u derived above applies for $0 < x < L$. For a solution to this equation to be determined, extra conditions on the solution u at each end are needed. These are called *boundary conditions*, and must be derived/modeled as

[1] The x-coordinate of any mass point on the string does not change in time.

well. Later on we will see that exactly one boundary condition (a restriction on u and its derivatives) is needed at each end, $x = 0$ and $x = L$.

Answer the following two extra questions:

(iv) What boundary condition applies at an end where **the string is tied?** (e.g., as in a guitar).

(v) What boundary condition applies at an end where **the string is free?**
For example, imagine a no-mass cart that can slide up and down (without friction) a vertical rod at $x = 0$, and tie the left end of the string to this cart. This keeps the left end of the string at $x = 0$, with the rod canceling the horizontal component of the tension force, but $u(0,\, t)$ can change.
A no-mass-no-friction cart is an idealization, approximating a setup where the mass and friction are small.

Remark 1.1 *Some details:*

#1 We idealize the string as a curve. This is justified as long as $\lambda \gg d$, where λ = scale over which motion occurs, and d = string diameter. This condition justifies item **3** above as well.

#2 The tension is generated by elastic forces (assume that the string is stretched). At a point along the string, the tension is the force with which each side (to the right or left of the point) pulls on the other side,[2] and it is directed along the direction tangent to the string.[3]

#3 At equilibrium the tension must be constant. For imagine that the tension is different at two points $a < b$ along the string. Then the segment $a \leq x \leq b$ would receive a net horizontal force (the difference in the tension values), and thus could not be at equilibrium.

#4 The hypothesis made imply that any changes in the string length can be neglected. Thus the amount of stretching, which generates the tension, does not change significantly anywhere. Hence the tension remains, essentially, equal to the equilibrium tension T. ♣

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2 DiAn01. Diffusion speed for ν constant (at what rate does an ink blob grow?)

Statement: Diffusion speed for ν constant

In the lectures it was shown that if $\boldsymbol{\Theta = \Theta(\vec{x},\, t)}$ denotes the concentration of salt in water (e.g.: grams per liter) then, **assuming that there is no motion** by the water

$$\Theta_t = \nu\, \Delta\, \Theta, \tag{2.1}$$

where $\boldsymbol{\nu}$ is the *diffusion coefficient* — which we assume here to be constant[4] — and Δ is the *Laplace operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The **same equation** (with a different value of ν) applies if, for example, Θ denotes sugar concentration, or the concentration of some coloring (e.g. ink).

Under the conditions where (2.1) applies, imagine that you inject a very tiny blob of ink inside the liquid.[5] Then the size of the blob will start increasing with time (due to the ink's diffusion). The blob's edge will cease to be sharp as time goes on, but here we will simplify things and assume that it remains sharp enough during the course of the experiment.

Make the approximation that, at time $t = 0$ (when you start the experiment) the blob is just a point. Then, *using qualitative and physical arguments only* **(do not attempt to solve the equation)**, *answer these questions:*

1. What are the dimensions of $\boldsymbol{\nu}$? Explain!

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[2] If you were to cut the string, this would be the force needed to keep the lips of the cut from separating.
[3] There are no normal forces, see item **3.**
[4] Generally ν need not be a constant; it may depend (for example) on the temperature.
[5] Carefully, so as not to start motion. The ink density must match that of water, to avoid gravity induced motion.

2. Argue that the shape of the blob is a sphere for $t > 0$.
3. Find a formula for the radius of the blob $R = R(t)$ as a function of time. There is a numerical multiplicative constant α, as in $\boldsymbol{R = \alpha\, f(t)}$, which you will not be able to determine (without solving the p.d.e.), but **you should be able to get $\boldsymbol{f(t)}$.**

Typical values for diffusion coefficients, as in the examples above, are $\boldsymbol{\nu = \gamma\, 10^{-5}\ \mathrm{cm}^2/\mathrm{sec}}$, where γ ranges [6] between $1/2$ and 2. **Assume that $\boldsymbol{\alpha = \gamma = 1}$.** Then

4. What is the radius of the blob when $t = 1$ minute?
5. At what time does $R = 5$ cm?

These numbers should give you an idea of how long it would take to sweeten a cup of coffee if you just deposited a lump of sugar in it, and did not stir the coffee.
Hint. *The radius $\boldsymbol{R}$ is a length; a function of time. However, the problem has only one dimensional parameter,*[7] $\boldsymbol{\nu}$. *How then do you get a length R from a time $\boldsymbol{t}$? There is only one way!*

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3 ExID51. Directional derivatives

Statement: Directional derivatives

Calculate the derivatives indicated below

1. Express $\dfrac{du}{ds}$ — for $u = u(x(s),\, y(s))$ — in terms of x, y, and u, given that:
   $x(s) = e^s$, $y(s) = s\, e^s$, and u satisfies $x\, u_x + (y + x)\, u_y = 2\, x + u$.
2. Express $\dfrac{du}{dt}$ — for $u = u\left(t^2,\, \ln(t),\, \dfrac{1}{t}\right)$ — in terms of the partial derivatives of $u = u(x,\, y,\, z)$ and t.
3. Let $u = e^x\, \sin(y)$, and let $(r,\, \theta)$ be the polar coordinate radius and angle. Calculate $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$ in terms of x and y.

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4 ExID03. Solve the one directional wave equation

Statement: Solve the one directional wave equation

Consider the one directional wave equation for $u = u(x,\, t)$, where c is a constant:

$$u_t + c u_x = 0. \tag{4.1}$$

Introduce the new independent variables $\boldsymbol{\eta = x - c\, t}$ and $\boldsymbol{\xi = x + c\, t}$, and change variables to write the equation for $\boldsymbol{u}$ as a function of these new variables: $\boldsymbol{u = u(\eta,\, \xi)}$. Using the transformed form of the equation, integrate it to show that it must be

$$u = f(\eta), \tag{4.2}$$

[6] Larger molecules diffuse slower than smaller ones; thus $\gamma(\mathrm{sugar}) < \gamma(\mathrm{salt})$.
Further, γ grows with temperature, and may even depend on the concentration.

[7] Note that at time $t = 0$ the blob has no dimension; it is a point!

for some arbitrary function f. Hence, the solutions to (4.1) must have the form $u = f(x - c\,t)$.

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5 TFPa02. Near constant initial data (linearize)

Statement: Near constant initial data (linearize)

Let

$$\boldsymbol{q = \frac{4\,q_m}{\rho_j^2}\,\rho\,(\rho_j - \rho)}, \quad (5.1)$$

where $\boldsymbol{q_m}$ is the maximum flow rate and $\boldsymbol{\rho_j}$ is the jamming concentration.

Let the concentration at time $t = 0$ be specified by

$$\boldsymbol{\rho(x,\,0) = \rho_0 + \epsilon\,\rho_j\,f(x)}, \quad (5.2)$$

where $\boldsymbol{0 < \rho_0 < \rho_j}$ is some constant density, $\boldsymbol{0 < \epsilon < (\rho_j - \rho_0)/\rho_j}$ is a numerical constant, and (for some constant distance $\boldsymbol{D > 0}$)

$$f(x) = \left\{ \begin{array}{ll} 0 & \text{for } |x| \geq D, \\ 1 - \dfrac{x^2}{D^2} & \text{for } |x| \leq D, \end{array} \right\}. \quad (5.3)$$

If $\boldsymbol{0 < \epsilon \ll 1}$ is small (e.g.: $\epsilon = 0.1$), **use linear theory to determine the concentration $\boldsymbol{\rho(x,\,t)}$ at a later time. In particular:** set $\boldsymbol{\rho_0 = 0.75\,\rho_j}$, and find the wave speed and draw in space–time (x–t plane) the characteristics marking the front and back of the disturbance. On the same graph, plot the trajectories of the vehicles (which move with the space average velocity u) whose initial positions are $x = 0$ and $x = 2\,D$.

Repeat the exercise for $\boldsymbol{\rho_0 = 0.25\,\rho_j}$ and $\boldsymbol{\rho_0 = 0.5\,\rho_j}$. **What is un-usual about the last case?**

For the purposes of the plotting, **assume space and time units where $\boldsymbol{D = 1}$** and $\boldsymbol{q_m = \rho_j}$ — **questions: is this possible? What are the units?**

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6 TFPa06. Solve a first order linear pde

Statement: Solve a first order linear pde

If $a \neq 0$, b, and c are constants, show that any function $u = u(x,\,y)$ that satisfies the partial differential equation

$$a\,u_x + b\,u_y = c\,u, \quad (6.1)$$

must be of the form

$$u = e^{c\,x/a}\,F(b\,x - a\,y), \quad (6.2)$$

where F is an arbitrary function.

Hint: *Use* $\xi = b\,x - a\,y$ *and* $\eta = x$ *as the new independent variables.*

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7 TFPa07. Solve a first order linear pde

Statement: Solve a first order linear pde

If c and β are a constants, find the solution to

$$u_t + c\,u_x = -\beta^2\,u, \quad (7.1)$$

with initial condition $u(x,\,0) = \sin x$.

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8 TFPa08. General solution 1-D wave equation

Statement: General solution 1-D wave equation

If $\xi = x - c\,t$ and $\eta = x + c\,t$, show that the wave equation

$$u_{tt} - c^2\,u_{xx} = 0 \quad (8.1)$$

transforms to $\boldsymbol{u_{\xi\,\eta} = 0}$. Then deduce that

$$u = F(x - c\,t) + G(x + c\,t) \quad (8.2)$$

is the general solution to (8.1), with F and G arbitrary functions.
Interpret this result in terms of traveling waves.
If $\boldsymbol{u(x,\,0) = \phi(x)}$ and $\boldsymbol{u_t(x,\,0) = \psi(x)}$,
show that

$$u = \frac{1}{2}\,\Big(\phi(x - c\,t) + \phi(x + c\,t)\Big) + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} \psi(\xi)\,\mathrm{d}\xi. \quad (8.3)$$

Given this formula, **describe (in words) what happens when $\boldsymbol{\psi = 0}$.**

THE END.