Problem Set # 01, 18.300 MIT (Spring 2022)

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Due: Fri February 18 (turn it in via the canvas course website).

Contents

1 Compute a channel flow rate function #01

Statement: Compute a channel flow rate function #01

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{1.1}$$

where $A = A(x,t)$ is the cross-sectional filled area of the river bed, x measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate q should be a function of A only [1] follows from the assumption of quasi-equilibrium. Then q is determined by a local balance between the friction forces and the force of gravity down the river bed.

[1] Possibly also x. That is: $q = Q(x, A)$, to account for non-uniformities along the river.

Assume now a man-made channel, with uniform triangular cross-section† and a uniform (small) downward slope, characterized by an angle θ. Assume also that the frictional forces are proportional to the product of the flow velocity u down the channel, and the wetted perimeter P_w of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function Q should have.** † Isosceles triangle, with bottom angle ϕ.

Hints: (1) $Q = u\,A$, *where* u *is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter* P_w *is proportional to some power of* A.

2 Conservation of probability in QM

Conservation of probability in QM

In non-relativistic quantum mechanics the motion of a point particle in a potential V is described by Schrödinger's equation.

$$i\,\hbar\,\psi_t = -\frac{\hbar^2}{2\,m}\,\psi_{xx} + V(x)\,\psi \quad \text{in 1D,} \tag{2.1}$$

where $\hbar = \frac{h}{2\,\pi}$ is the Plank constant divided by $2\,\pi$,
$\psi = \psi(x,\,t)$ is the (*complex valued*) wave function,
m the particle's mass, and i is the imaginary unit.
The interpretation is that [2]

$$\tilde{\rho} = |\psi|^2 = \psi\,\psi^* \tag{2.2}$$

is the pdf [probability distribution function] (pdf)
for the particle position. That is, the probability
of finding the particle in any interval $a < x < b$ is [3]

$$\int_a^b \tilde{\rho}\,dx. \tag{2.3}$$

Now: probability is conserved, and $\tilde{\rho}$ is its density. **Question: What is the probability flux?**
Hint. Use (2.1) to find an equation of the form $\tilde{\rho}_t + \tilde{q}_x = 0$. *The flux is then* $\tilde{q}$.
Warning: check that the flux you obtain is real valued.

3 Dispersive Waves and Modulation

Statement: Dispersive Waves and Modulation

Consider the following linear partial differential equations for the scalar function $\boldsymbol{u = u(x,\,t)}$:

$$\begin{aligned} \boldsymbol{u_t + c\,u_x + d\,u_{xxx}} &= \boldsymbol{0}, && (3.1)\\ \boldsymbol{u_{tt} - u_{xx} + a\,u} &= \boldsymbol{0}, && (3.2)\\ \boldsymbol{i\,u_t + b\,u + g\,u_{xx}} &= \boldsymbol{0}, && (3.3) \end{aligned}$$

where the equations are written in a-dimensional variables, $\boldsymbol{(c,\,d,\,a,\,b,\,g)}$ are real constants, and $\boldsymbol{a > 0}$. These equations arise in many applications, but we will not be concerned with them here. It should be clear that, in all three cases,

$$\boldsymbol{u = A\,e^{i\,(k\,x - \omega\,t + \theta_0)}}, \quad \text{where} \quad \boldsymbol{\omega = \Omega(k)}, \tag{3.4}$$

is a solution of the equations, for any real constants $A > 0$, θ_0, k, and ω, provided that

[2] Here * indicates the complex conjugate.
[3] ψ should be normalized so that $\int \tilde{\rho}\,dx = 1$, where the integral is over the whole domain where the particle resides.
The units for ψ are $1/\sqrt{\text{length}}$ in 1D.

M1. For equation (3.1): $\Omega(k) = c\,k - d\,k^3$ .... **Verify that this is true.**

M2. For equation (3.2): $\Omega(k) = \pm\sqrt{a + k^2}$ .... **Verify that this is true.**

M3. For equation (3.3): $\Omega(k) = -b + g\,k^2$ .... **Verify that this is true.**

Note that the general solution to the equations can be written as a linear combination of solutions of this type, via **Fourier Series** and **Fourier Transforms** — we will see this later in the semester.

Remark 3.1 *Solutions such as that in (3.4) represent monochromatic sinusoidal traveling waves, with* **amplitude** A, **phase** $\theta = k\,x - \omega\,t + \theta_0$, **wave number** k, *and* **angular frequency** ω. *The* **wave length** *and* **wave period** *are* $\lambda = 2\,\pi/k$ *and* $\tau = 2\,\pi/\omega$, *respectively. The wave profile's crests and troughs move at the speed given by* $\theta =$ *constant, namely: the* **phase speed** $c_p = \omega/k$. ♣

Remark 3.2 *In all three cases,* $\Omega = \Omega(k)$ **is a real valued function of** k**, with** $\frac{d^2\Omega}{dk^2} \neq 0$ — *i.e.:* Ω *is* **not** *a linear function of* k. *Because of this, we say that the* **equations are dispersive** *and call* Ω **the dispersion function.** *The (non-constant) velocity* $c_g = c_g(k) = d\omega/dk$ *is called the* **group speed**, *and the objective of this problem is to find out what the meaning of* c_g *is.*

The reason for the name "dispersive" is as follows: In a dispersive system, waves with different wavelengths propagate at different speeds. Thus, a localized initial disturbance, made up of many modes of different wavelengths, will **disperse** *in time, as the waves cease to add up in the proper phases to guarantee a localized solution. This is because localization depends on destructive interference, outside some small region, of all the modes* $a(k)\,e^{i\,(k\,x+\theta_0)}$ *making up the initial disturbance. However, since these modes propagate at different speeds, the phase coherence needed for destructive interference is destroyed by the time evolution.* **This phenomena is illustrated in figure** 3.1. ♣

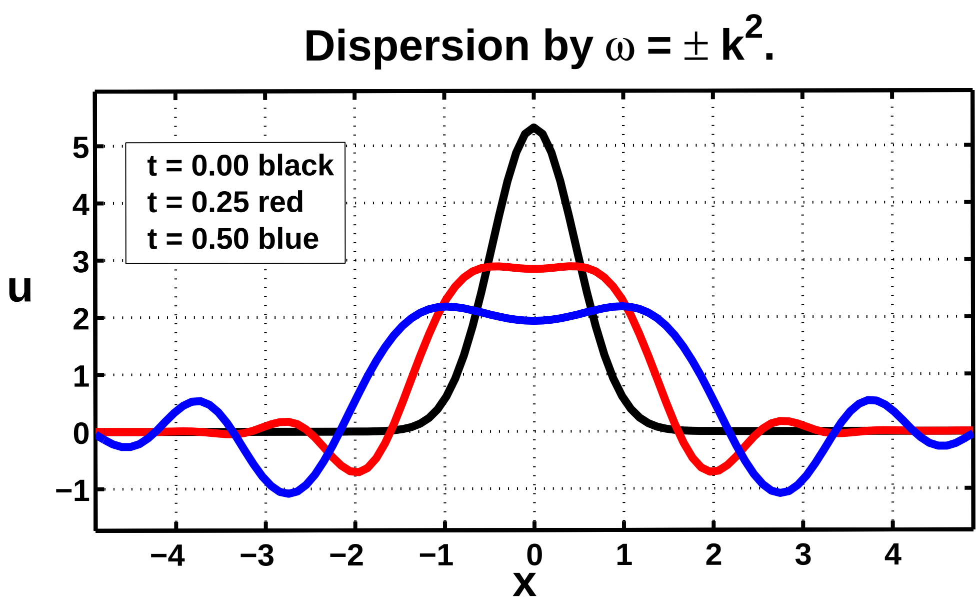

Figure 3.1: Example of dispersion: initial "Gaussian" bump, as it evolves under a dispersive equation with $\Omega(k) = \pm k^2$ — i.e.: $u_{tt} + u_{xxxx} = 0$. The solution at times $t = 0, 1/4, 1/2$ displayed. As the initial lump's phase coherence is destroyed by dispersion, localization is lost, and the bump "disperses". The solution $u = u(x,\,t)$ is given by the Fourier integral $u = \text{Re}\left(\int_{-\infty}^{\infty} a(k) e^{i(k\,x - k^2\,t)} \mathrm{d}k\right)$, where $a(k) = e^{-k^2/9}$.

The tasks to be performed

TASK 1. verify **M1** through **M3**, above below equation (3.4).

TASK 2: Consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number k and angular frequency ω are related by a dispersion relation

$$\omega = \Omega(k). \tag{3.5}$$

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a *local wave number* $\boldsymbol{k = k(x,\ t)}$ and a *local angular frequency* $\boldsymbol{\omega = \omega(x,\ t)}$. In particular, assume that both k and ω vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods — *on the other hand, they may change considerably over many wave lengths or wave periods.* Then

Assuming conservation of wave crests, derive equations governing k and ω.

These equations are called the **Wave Modulation Equations.**

Remark 3.3 *The assumption that k and ω vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.* ♣

Remark 3.4 **Why is it reasonable to assume that the wave crests are conserved?** *The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where the assumption of slow variation is invalid. There are some pretty interesting research problems in pattern formation that are related to this point.* ♣

Hint 3.1 *It should be clear that one of the equations is $\omega = \Omega(k)$, since the solution behaves locally like a monochromatic wave* (this is the "quasi-equilibrium" approximation in this context). *For the second equation, express the density of wave crests (and its flux) in terms of k and ω. To figure this out, think of the following questions* (i) How many wave crests are there per unit length for a sinusoidal wave? (ii) How many wave crests pass through a fixed point in space, per unit time, for a sinusoidal wave? *Then write the equation for the conservation of wave crests using these quantities.* ♣

4 Fundamental Diagram of Traffic Flow #01

Statement: Fundamental Diagram of Traffic Flow #01

The desired car velocity $u = U(\rho)$ has its maximum, u_m, at $\rho = 0$, and vanishes at the jamming density, ρ_J. Assuming that U is a linear function of ρ, write a formula for the flow rate $q = Q(\rho)$. What is the road capacity q_m? What is the wave velocity $c = c(\rho)$?

5 Fundamental Diagram of Traffic Flow #03

Statement: Fundamental Diagram of Traffic Flow #03

Many state laws state that: for each 10 mph (16 kph) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 ft (5 m)). There is another law that gives a maximum speed limit (assume that this is 50 mph (80 kph)). **Find the flow of cars as a function of density, $q = q(\rho)$**, that results from these two laws.

The state laws on following distances stated in the prior paragraph were developed in order to prescribe a spacing between cars such that rear-end collisions could be avoided, as follows:

a. Assume that a car stops instantaneously. *How far would the car following it travel* if moving at u mph and
 a1. The driver's reaction time is τ, and
 a2. After a delay τ, the car slows down at a constant maximum deceleration α.

b. The calculation in part **a** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate α, but the driver in the following car still takes a time τ to react. *How far back does a car have to be, traveling at u mph, in order to prevent a rear-end collision?*

c. Show that the law described in the first paragraph of this problem corresponds to part **b**, if the human reaction time is about 1 sec. and the length of a car is about 16 ft (5 m).

Note: *What part* **c** *is asking you to do is to justify/derive the state law prescription, using the calculations in part* **b** *to arrive at the minimum car-to-car separation needed to avoid a collision when the cars are forced to brake.*

Practice problems

The problems below are practice problems. These are things important for what follows in the course, and which you likely know, but may need a bit of practice on.

6 ExID03. Single variable implicit differentiation

Statement: Single variable implicit differentiation

In each case compute $y' = \dfrac{dy}{dp}$ as a function of y and p, given that $y = y(p)$ satisfies:

1. $p^3 + p\,y + 2 = 0$.
2. $y = \sin(y + p)$.
3. $\ln(y) = p$.
4. $\cos^2(y) = p$, for $p > 0$.
5. $y = f(c - y\,p)$.
6. $y = f(p - c\,y)$.

Note: in (5) and (6) f is an arbitrary function, and c is a constant.

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7 ExID14. Two variable implicit differentiation

Statement: Two variable implicit differentiation

In each case compute $u_x = \dfrac{\partial u}{\partial x}$ and $u_p = \dfrac{\partial u}{\partial p}$ (as functions of u, x, and p), given that $u = u(x,\,p)$ satisfies:

1. $\cos(p^2\,u) = p\,e^{-x^2}$.
2. $p = \cos(x + u)$.
3. $u = p\,f(x + u)$.

Note: In (3) f is an arbitrary function of a single variable, $f = f(\zeta)$.

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8 ExID42. Differentiation within integrals

Statement: Differentiation within integrals

In each case compute $u_x = \dfrac{\partial u}{\partial x}$ and $u_p = \dfrac{\partial u}{\partial p}$ (as functions of u, x, and p), given that $u = u(x,\, p)$ satisfies:

1. $p = \displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,ds.$

2. $u = \displaystyle\int_0^x \sin\left(p\,u(s^2,\, s) + x\,s\right)\,ds.$

3. $p = \displaystyle\int_x^u \cos\left(p\,\sin(s) + x\,s^2\right)\,ds.$

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9 ExID56. Directional derivatives and Taylor

Statement: Directional derivatives and Taylor

Do the tasks stated in items **1** and **2** below

1. Let Γ be a curve in the plane, $\vec{r} = (x,\, y)$, parameterized by arc-length: $x = X(s)$ and $y = Y(s)$. Assume that $\boldsymbol{\frac{dY}{ds} < 0}$ along the curve, and that the curve is tangent to the unit circle for $s = 0$, at the point $(x,\, y) = (1/\sqrt{2},\, 1/\sqrt{2})$.
Calculate $\boldsymbol{\frac{d\Phi}{ds}}$ *at* $s = 0$*, along the curve* Γ*, for* $\boldsymbol{\Phi = \sin\left(\frac{\pi}{\sqrt{2}}\,x + \pi\,y^2\right)}$.
Correct answer required. "I only missed a sign", or similar, excuses not allowed. **Check** your answer!

2. Let Γ be the straight line in the plane, $\vec{r} = (x,\, y)$, given by $x = 1 + t$ and $y = t$, $-\infty < t < \infty$. Let $\Phi = \Phi(\vec{r})$ be some smooth scalar function. Define $f = f(t)$ by $f = \Phi$ along Γ.
Write the first three terms of the Taylor expansion for f *at* $t = 0$*, in terms of the partial derivatives of* Φ *at* $\vec{r}_0 = (1,\, 0)$*. In particular, compute* $\dot{f}(0)$ *and* $\ddot{f}(0)$ *for* $\Phi = x^2\,e^y$*.*

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10 ExID61. Direct Taylor expansions

Statement: Direct Taylor expansions

For the examples below, calculate the Taylor expansion up to the order indicated (e.g.: $\cos(x) = 1 - \frac{1}{2}\,x^2 + O(x^4)$). Do not use a calculator to evaluate constants that appear in the expansions — e.g., $\sqrt{2}/\pi$ or $\cos(3)$. On the other hand, do simplify when possible — e.g., $\tan(\pi/4) = 1$ or $2/\sqrt{2} = \sqrt{2}$.

1. Expand, up to $O(x^4)$, $f(x) = \sin(x)\,\cos(\sqrt{x})$.

2. Expand, up to $O(x^5)$, $f(x) = \sin(1 + x)$.

3. Expand, up to $O(x^5)$, $f(x) = \sin(1 + x + x^3)$.

4. Let $G = G(x,\, y)$ be some smooth[4] function of two variables. For $z \geq 0$, expand up to $O(z^3)$, $f(z) = G(z,\, z^{1.5})$. Express the expansion coefficients in terms of partial derivatives of G.

[4] The partial derivatives of f, to any order, exist and are continuous

11 ExID71. Change of variables for an ode

Statement: Change of variables for an ode

Consider the second order, nonlinear, ode

$$x^2 \cos w \, \frac{d^2 w}{dx^2} - x^2 \sin w \left(\frac{dw}{dx}\right)^2 + x \cos w \, \frac{dw}{dx} + \sin w = 0 \tag{11.1}$$

for $w = w(x)$, where $x > 0$. Rewrite it in terms of $u = u(y)$, where $u = \sin w$ and $y = \ln x$.

12 ExID77. Change of variables for a pde

Statement: Change of variables for a pde

Let $u = u(x,\, t)$ be a solution of the heat

$$u_t = u_{xx}. \tag{12.1}$$

What equation does $\phi = -\frac{1}{u}\, u_x$ satisfy?

Hint. Calculate ϕ_t and use the equation for u. Calculate ϕ_x and write it in terms of u, u_{xx}, and ϕ^2. Then compute ϕ_{xx}. You should now be able to write ϕ_t in terms of ϕ, ϕ_x, and ϕ_{xx}.

THE END.