Answers to P-Set # 08, 18.300 MIT (Spring 2022)

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1 Introduction. Experimenting with Numerical Schemes

Consider the numerical schemes that follow after this introduction, for the specified equations. **YOUR TASK here is to** *experiment (numerically) with them so as to answer the question:* **Are they sensible?** Specifically:

- **i.1** Which schemes give rise to the type of behavior illustrated by the "bad" scheme in the GBNS_lecture script?[1] of the 18.311 MatLab Toolkit?
- **i.2** Which ones behave properly as Δx and Δt vanish?

Further: **show that** they all arise from some approximation of the derivatives in the equations, similar to the approximations used to derive the "good" and "bad" schemes used in the GBNS_lecture script[2] of the 18.311 MatLab Toolkit. That is: **show that the schemes are consistent.**‡

‡ I **strongly recommend** that you read the *Stability of Numerical Schemes for PDE's* notes in the course WEB page **before** you do these problems.

Remark 1.1 *Some of the schemes are "good" and some are not. For the "good" schemes restrictions are needed (as Δx gets small) on Δt to avoid bad behavior — i.e.: fast growth of grid-scale oscillations. Specifically:* **in all the schemes a parameter appears:** $\boldsymbol{\lambda = \Delta t/\Delta x}$ *in some cases, and* $\boldsymbol{\nu = \Delta t/(\Delta x)^2}$ *in others. You* **will need to keep this parameter smaller than some constant** *to get the "good" schemes to behave. That is:* $\boldsymbol{\lambda < \lambda_c}$*, or* $\boldsymbol{\nu < \nu_c}$*.* **For the "bad" schemes, it will not matter how small λ (or ν) is. Figuring out the values of these constants is also part of the problem.** *For the assigned schemes the constants, when they exist ("good" schemes), are* **simple $O(1)$ numbers***, somewhere between 1/4 and 2. That is, stuff like 1/2, 2/3, 1, 3/2, etc. —* **not** *things like $\lambda_c = \pi/4$ or $\nu_c = \sqrt{e}$.* **You should be able to find them** *by careful numerical experimentation.* ♣

[1] Alternatively: check the *Stability of Numerical Schemes for PDE's* notes in the course WEB page.
[2] Alternatively: check the *Stability of Numerical Schemes for PDE's* notes in the course WEB page.

Remark 1.2 *In order to do these problems you may need to write your own programs. If you choose to use MatLab for this purpose, there are several scripts in the 18.311 MatLab Toolkit that can easily be adapted for this. The relevant scripts are:*

- *The schemes used by the* GBNS_lecture *MatLab script are implemented in the script* InitGBNS.
- *The two script series*
  PS311_Scheme_A, PS311_Scheme_B, ... *and* PS311_SchemeDIC_A, PS311_SchemeDIC_B, ...,
  have several examples of schemes already setup in an easy to use format. Most of the algorithms here are already implemented in these scripts. The scripts are written so that modifying them to use with a different scheme involves editing only a few (clearly indicated) lines of the code.
- *Note that the scripts in the 18.311 MatLab Toolkit are written avoiding the use of "for" loops and making use of the vector forms MatLab allows — things run a lot faster this way. Do your programs this way too, it is good practice.* ♣

Remark 1.3 **Do not** *include lots of graphs and numerical output in your answers. Explain what you did and how you arrived at your conclusions, and illustrate your points with a few selected graphs.* ♣

Remark 1.4 **In all cases the notation:** $x_n = x_0 + n\Delta x$, $t_k = t_0 + k\Delta t$, **and** $u_n^k = u(x_n, t_k)$, **is used.** ♣

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2 GBNS02. scheme B. Backward differences for $u_t + u_x = 0$

2.1 Statement: Scheme B. Backward differences for $u_t + u_x = 0$

Equation: $u_t + u_x = 0$. Scheme: $u_n^{k+1} = u_n^k - \lambda\,(u_n^k - u_{n-1}^k)$, where $\lambda = \dfrac{\Delta t}{\Delta x}$.

Reminder/read the introduction! Here you are asked to (numerically) study this scheme and decide if it is unstable or not. Specifically, for this scheme:
Is there some positive value of λ, $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, the solution of the scheme converges to the solution to the PDE as $\Delta x \to 0$? In this case **the scheme is stable.**
Else, for any $\lambda > 0$, as $\Delta x \to 0$, the solutions develop very large amplitude grid scale oscillations. Then **the scheme is unstable.**
If there is a λ_c, find it (approximately) by experimenting (numerically) with the scheme.
WARNING: even a stable scheme blows up if $\lambda > \lambda_c$. Observing blow up for one λ is not enough to conclude instability. This does not mean you have to check arbitrarily small λ's — typically $\lambda_c = O(1)$. Note also that, to observe the blow up you need to run many time steps. As $\Delta x \to 0$, keep the time interval over which you solve the equation fixed, say: from $t = 0$ to $t = 1$.

2.2 Answer: Scheme B. Backward differences for $u_t + u_x = 0$

This is the scheme implemented by the PS311_Scheme_B and the PS311_SchemeDIC_B scripts in the 18311 MatLab Toolkit. You should have found that this **scheme is stable:** **for $\lambda < \lambda_c = 1$ the numerical scheme does not amplify the noise.** This **scheme is also consistent.** Consistency follows because the scheme equations can be written in the form

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} + \frac{u_n^k - u_{n-1}^k}{\Delta x} = 0,$$

which are satisfied with errors of $O(\Delta t,\, \Delta x)$ by any smooth solution of the equation $u_t + u_x = 0$.

Note that, as a **consequence of stability and consistency, the scheme is convergent:** *as $\Delta x \to 0$ the numerical solution converges to the exact solution, provided that $\lambda < \lambda_c$.*

3 GBNS03 scheme C. Centered differences for $u_t + u_x = 0$

3.1 Statement: Scheme C. Centered differences for $u_t + u_x = 0$

Equation: $u_t + u_x = 0$. Scheme: $u_n^{k+1} = u_n^k - \frac{1}{2}\lambda\,(u_{n+1}^k - u_{n-1}^k)$, where $\lambda = \dfrac{\Delta t}{\Delta x}$.

Reminder/read the introduction! Here you are asked to (numerically) study this scheme and decide if it is unstable or not. Specifically, for this scheme:
Is there some positive value of λ, $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, the solution of the scheme converges to the solution to the PDE as $\Delta x \to 0$? In this case **the scheme is stable.**
Else, for any $\lambda > 0$, as $\Delta x \to 0$, the solutions develop very large amplitude grid scale oscillations. Then **the scheme is unstable.**
If there is a λ_c, find it (approximately) by experimenting (numerically) with the scheme.
WARNING: even a stable scheme blows up if $\lambda > \lambda_c$. Observing blow up for one λ is not enough to conclude instability. This does not mean you have to check arbitrarily small λ's — typically $\lambda_c = O(1)$. Note also that, to observe the blow up you need to run many time steps. As $\Delta x \to 0$, keep the time interval over which you solve the equation fixed, say: from $t = 0$ to $t = 1$.

3.2 Answer: Scheme C. Centered differences for $u_t + u_x = 0$

This is the scheme implemented by the PS311_Scheme_C and the PS311_SchemeDIC_C scripts in the 18311 MatLab Toolkit. You should have found that this **scheme is unstable: there is no value $\lambda_c > 0$ such that (for $\lambda < \lambda_c$) the numerical solution does not develop exponentially large, grid scale, oscillations as $\Delta x \to 0$.**

On the other hand, the **scheme is consistent.** It's equations can be written in the form

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} + \frac{u_{n+1}^k - u_{n-1}^k}{2\Delta x} = 0,$$

which are satisfied with errors of $O(\Delta t,\, (\Delta x)^2)$ by any smooth solution of the equation $u_t + u_x = 0$.

4 Experiments with a slinky

4.1 Statement: Experiments with a slinky

Consider a homogeneous cylindrical rod (made of an elastic material), subject to (small amplitude) longitudinal deformations. Let x be the length coordinate (measured along the axis of the cylinder) when the cylinder is in its relaxed position. Use x as a label for the mass elements in the cylinder.[3] For every mass element x, let $u = u(x,\, t)$ be its position at time t, measured along the axis of the cylinder (note that $u = x$ corresponds to the cylinder at rest.) Then u describes the state of the cylinder at any time and obeys the **wave equation:**

$$u_{tt} - c^2\, u_{xx} = 0, \quad \text{where} \quad c = \sqrt{k/\rho}. \tag{4.1}$$

[3] Since we are considering only longitudinal motions, points in a cross section move in unison and we need not label them separately.

Here $\boldsymbol{\rho}$ is the density (mass per unit length) of the rod, and $\boldsymbol{k}$ characterizes the elastic response of the material: if we stretch the cylinder by an amount ΔL, then the elastic force is $k\dfrac{\Delta L}{L}$, where L is the length of the rod (note that $\boldsymbol{k}$ has the dimensions of a force, thus c is a speed).

Remark 4.1 *The basic assumption here is that the cylinder remains at all times within the regime of applicability of Hooke's law. This means that the deformations (given by $u_x - 1$) are small enough everywhere. In particular, this also implies that variations in the cross-section of of the cylinder can be ignored (e.g.: if volume is preserved, the cross section will be larger in regions under compression than in those under tension).*

In the derivation of equation (4.1) it is assumed that the elastic forces are dominant, so that other forces (e.g.: gravity) can be ignored. For a rod with a vertical orientation, such that the elastic forces are not dominant over gravity, equation (4.1) must be modified to:

$$u_{tt} - c^2\, u_{xx} = -g, \tag{4.2}$$

where $\boldsymbol{g}$ is the acceleration of gravity, and we assume that the vertical coordinate $\boldsymbol{x}$ increases upwards. In particular, **consider the case of a rod hanging vertically without any motion** (i.e.: $u = u(x)$, with no time dependence), and measure $\boldsymbol{x}$ from the bottom of the rod. Then:

$$u = 0 \quad \text{and} \quad u_x = 1 \;\text{ at }\; x = 0, \tag{4.3}$$

where the second equation follows because there is no force at the lower end (no section of the rod below that must be supported). Then the equation for $u = u(x)$, namely:

$$c^2\, u_{x\,x} = g,$$

can be integrated to yield:

$$\boldsymbol{u = \frac{g}{2c^2}\, x^2 + x.} \tag{4.4}$$

A particular **example where this should apply to is that of a slinky. One objective of this problem is for you to check how well a slinky obeys equation (4.4).**

Proceed as follows:

1. Get a slinky in good condition and draw a straight line along its edge, parallel to the slinky's axis. Draw the line so that, when it reaches one end (the "bottom" end), it does so **at the end** of the coil that makes the slinky — i.e.: no more coil beyond the mark.

2. Starting from the "bottom" end of the slinky, name the points at which each coil is marked by the line as $n = 0, 1, \ldots$ Then (if $\boldsymbol{w}$ is the width of a coil) when the slinky is at rest, the position of the $\boldsymbol{n^{th}}$ point is given by

$$x_n = n\, w.$$

3. To find w, measure the total length of the slinky, and divide this by the number of coils. You can also easily measure the "density" ρ of the coil by weighting it and dividing the result by its length.

4. Hang the slinky in a vertical position [4] (with the bottom end down) and wait till it is at rest. Then measure the distance u_n of the n^{th} point from the point $n = 0$ at the bottom. One way to do this is to have a measuring tape on a wall right behind the hanging slinky.

5. Equation (4.4) predicts that

$$u_n = \frac{g}{2c^2}\, x_n^2 + x_n = \frac{g\rho}{2k}\, x_n^2 + x_n. \tag{4.5}$$

[4] For example, staple it to the underside of a shelf by a wall.

6. **The question is now:** How well does equation (4.5) match your measurements? Of course, you do not have k, but you will have several values of n. If (4.5) applies, then

$$u_{n+1} - u_n = \Delta u_n = \frac{g\,\rho}{k}\,w^2\,n + \frac{g\,\rho}{2\,k}\,w^2 + w.$$

Thus a plot of $\boldsymbol{u_{n+1} - u_n}$ versus $\boldsymbol{n}$ should give a straight line with slope $\boldsymbol{g\,\rho\,w^2/k}$. From this you can get $\boldsymbol{k}$, which is the hardest quantity to measure directly in this context.

Next suspend the slinky from one end and set it to vibrate (longitudinally). In this case, if we set the origin for the coordinate x at the top (where the slinky does not move), the governing equation will still be (4.2), but the boundary conditions are now:

$$u(0,\,t) = 0 \quad \text{and} \quad u_x(-L,\,t) = 1, \tag{4.6}$$

where $\boldsymbol{L}$ **is the length of the slinky in its rest state.** The first condition here follows because $x = 0$ corresponds to the clamped end at the top, while the second simply states that there are no elastic forces at the bottom end (same reason used when deriving the second condition in (4.3)). It is easy to see that these conditions (and the equation) are satisfied by the function

$$u = a\sin\left(\frac{\pi}{2\,L}\,x\right)\sin\left(\frac{c\,\pi}{2\,L}\,t\right) + \frac{g}{2\,c^2}\,x^2 + \left(1 + \frac{g\,L}{c^2}\right)\,x, \tag{4.7}$$

where a is an arbitrary constant. This solution corresponds to an oscillation with **period** $\boldsymbol{T = \dfrac{4\,L}{c}}$**.**

Now, **continue the experiment:**

7. In the situation described in the paragraph above, measure the period of the slinky — do not try to measure a single period, time several and then divide by the number of periods timed.
8. Compare the result of your measurement of the period with the one given by the formula for T above — from the prior steps you can obtain a value for c.
9. Discuss the results of your experiment.

4.2 Answer: Experiments with a slinky

These is the data obtained with one slinky, doing the measurements in rather primitive (home) conditions. The slinky basic characteristics were:

- Number of coils in the slinky .... $N =$ 40.
- Length of slinky when contracted .... $L =$ 6.59 cm.
- Width of each slinky coil ($w = L/N$) .... $w =$ 1.65 mm.
- Total mass of slinky .... $M =$ 86.4 gr.
- Density $\rho = M/L$ of slinky .... $\rho =$ 13.1 gr/cm.

Note 1: You should not trust the third number in these quantities too much. The error in L is in the order of 0.5 mm at best, for example (a carpenter tape, with the smallest division a 1/16 of an inch, was used to do length measurements). As we will see below, we will not need any of these quantities (except for the number of coils $\boldsymbol{N}$) to check the validity of our "slinky theory".

Note 2: An important point is that, when suspending the slinky the last two coils were used. Thus, we **take $\boldsymbol{N = 39}$ and $\boldsymbol{L = 6.42}$ cm in the calculations below.**

The distances between successive coils of the slinky, when hanging vertically without moving, numbered from bottom to top, are given below in equation (4.8). Again the measurements were made using a carpenter tape and then transformed into centimeters. Because it is rather hard to make these measurements with the slinky hanging, we should expect much less precision here than when measuring L above — in the order of millimeters: the second decimal digit below is probably not very meaningful, even the first is suspect.

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}
0.16	0.24	0.40	0.48	0.64	0.95	0.95	0.95	1.19	1.43	1.59	1.75	1.98
d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}
2.22	2.38	2.54	2.54	2.86	3.02	3.33	3.49	3.49	3.49	3.65	4.13	3.65
d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}	d_{37}	d_{38}	
3.81	3.81	3.97	4.13	4.29	4.45	4.45	4.60	4.76	5.08	5.40	6.35	

(4.8)

A plot of these distances versus n is shown in figure 4.1. The **theory predicts** that the points $(d_n,\, n)$ should be on

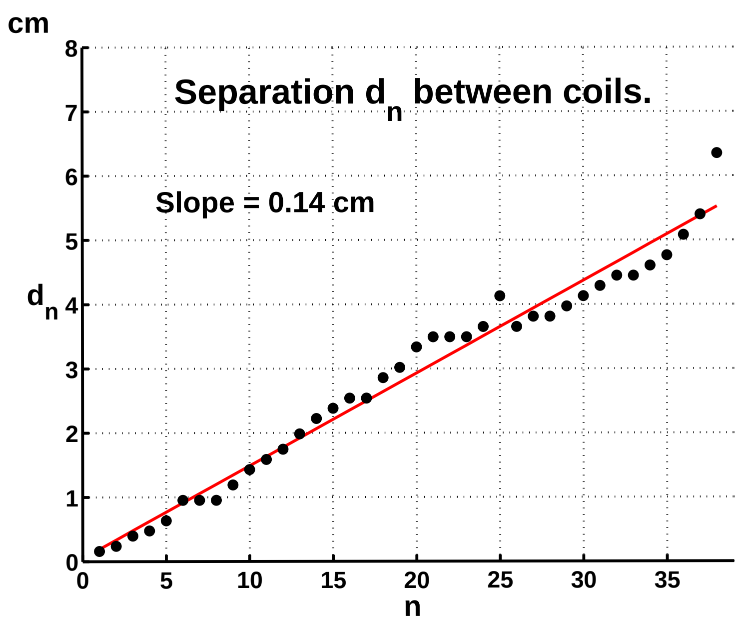

Figure 4.1: Experiments with a slinky. Slinky coil separation $\boldsymbol{d_n}$ as a function of $\boldsymbol{n}$, when the slinky is hanging vertically without moving (the index $\boldsymbol{n}$ is ordered from the bottom up). The theory predicts that this should be a straight line with slope $\boldsymbol{a = g\rho w^2 k^{-1}}$.

a line with slope $\boldsymbol{a = g\,\rho\,w^2\,k^{-1}}$, where $\boldsymbol{k}$ characterizes the elastic response of the slinky and $\boldsymbol{g} = 9.81$ cm/sec is the acceleration of gravity. From the data above we can get a least squares approximation to this slope, namely $\boldsymbol{a} = 0.14$ cm. The **theory also predicts** a value for the period of oscillation of the slinky (when hanging vertically), given by:

$$T_{\text{theory}} = \frac{4\,L}{c} = 4\,L\sqrt{\frac{\rho}{k}} = 4\,L\sqrt{\frac{a}{g\,w^2}} = 4\,\frac{L}{w}\sqrt{\frac{a}{g}} = 4\,N\sqrt{\frac{a}{g}} \approx 1.89 \text{ sec.}$$

To calculate this period only the slope $a = 0.14$ cm and the number of coils $N = 39$ are needed. None of the other parameters listed at the start of this problem answer is needed.

A direct measurement of the period went as follows:

- 1-st measurement: 4 oscillations in 6.5 sec.
- 2-nd measurement: 4 oscillations in 6.7 sec.

} which yields $T_{\text{exper}} = 1.65$ sec.

The two numbers agree to within 14 %, which is not great but not too bad either.

Discussion.

The agreement between the data in equation (4.8) and a straight line, as shown by figure 4.1, is quite reasonable (but not perfect). Actually, a calculation of the standard deviation of the measured d_n (from the ones given by the least squares straight line fit) yields $\sigma = 2.53$ mm — quite in line with what one should expect, given the crude conditions in which the experiment was conducted. Given that the average coil separation was $\mathrm{Mean}(d_n) = 2.86$ cm, the given σ represents an error in measuring the values d_n of size 10 %. Thus, we expect the value of a calculated from this to be not much better than this. Given that measuring the period is also quite hard (figuring out when a period ends/starts is not easy), the discrepancy between the measured period and the calculated period is quite in line with what should be expected.

Further observations:

- The value for d_{38} has a large error (furthest away from the straight line). Clearly this is one of the hardest to measure, since one of the two coils it involves is the last (tied) one.
- At some places the d_n line up, for three successive n's or so, horizontally. This is caused by round off to the nearest mark in the measuring tape, when the differences in length are small.
- I expected the largest errors in d_n to occur near the bottom, where the distances are small. But this turned out not to be true (at least for this experiment).
- A slinky is not really a "rod", so using the rod equations to describe its behavior involves a fairly rough approximation. It is not clear how much of the discrepancies in this experiment are due to sloppy experimental technique and how much is due to model imperfections. On the other hand, this simple example shows that even rough approximations can be quite good.
- The largest discrepancy appears to be between T_{theory} and T_{exper}. Here we would like to point out that the theoretical value arises, in particular, from assuming that the slinky is "perfectly" immovable at the top. However, it turns out that the period can be quite sensitive to what happens there. For example, suppose that there is some "elastic yield" at the point where the coil is suspended. This means that the condition $u(0,\, t) = 0$ in (4.6) has to be replaced by a condition balancing the slinky tensional force (given by $k(u_x - 1)$) and the restoring force trying to keep u vanishing there. Namely, we would end up with something like

  $$k\,(u_x - 1) = \alpha u \quad \text{for} \quad x = 0,$$

  where α is the elastic constant for the slinky support. In this case we must replace (4.7) by

  $$u = a\cos(\beta(x+L))\cos(c\,\beta\,t) + \frac{g}{2\,c^2}\,x^2 + \left(1 + \frac{g\,L}{c^2}\right)\,x + \frac{g\,k\,L}{\alpha\,c^2},$$

  where a is an arbitrary constant and β must satisfy $\boldsymbol{\alpha \cos(\beta L) + k\,\beta\,\sin(\beta L) = 0}$. The period is now $T = \dfrac{2\,\pi}{c\,\beta}$. Only for $\alpha = \infty$ does the prior result follow! For α large (but finite), a first correction to the $\alpha = \infty$ result yields:

  $$\beta \approx \frac{\pi}{2\,L}\left(1 + \frac{k}{\alpha\,L}\right) \quad\Longrightarrow\quad T \approx \frac{4\,L}{c}\left(1 - \frac{k}{\alpha\,L}\right).$$

  This is the sort of thing you would have to face if you merely use a piece of tape (or glue) to keep the slinky fixed at the top. In this case it would be very hard to estimate the size of α, but it is unlikely that the correction would be negligible. Other sloppy ways of suspending the slinky would have similar problems.

THE END.