Answers to P-Set # 07, 18.300 MIT (Spring 2022)

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1.1 Statement: Computer Exercise in Fourier Series

1.1.1 Introduction: Fourier Series

Generally, a 2π-periodic function $F = F(x)$ can be expressed in terms of its **Fourier Series** where the n^{th} complex Fourier coefficient F_n is defined by

$$F(x) = \sum_{n=-\infty}^{\infty} F_n\, e^{i\,n\,x}, \qquad (1.1)$$

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\, e^{-i\,n\,x}\, dx, \qquad n = 0,\ \pm 1,\ \pm 2,\ \pm 3, \dots \qquad (1.2)$$

An alternative formulation, obtained upon using $e^{-i\,n\,x} = \cos(n\,x) - i\,\sin(n\,x)$, is given by: where: (i) $c_0 = F_0$ and $c_n = (F_n + F_{-n})$ are

$$F(x) = c_0 + \sum_{n=1}^{\infty} (c_n\,\cos(n\,x) + s_n\,\sin(n\,x)), \qquad (1.3)$$

the cosine Fourier coefficients, and (ii) $s_n = i\,(F_n - F_{-n})$ are the sine Fourier coefficients. Thus

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx, \tag{1.4}$$

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,\cos(n\,x)\,dx, \quad \text{for: } n = 1,\,2,\,3,\,\ldots \tag{1.5}$$

$$s_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,\sin(n\,x)\,dx, \quad \text{for: } n = 1,\,2,\,3,\,\ldots \tag{1.6}$$

If F is real valued, then F_{-n} is the complex conjugate of F_n, so that

$$c_n = 2\,\mathrm{Re}\,(F_n) \quad \text{and} \quad s_n = -2\,\mathrm{Im}\,(F_n) \quad \text{for} \quad n > 0. \tag{1.7}$$

Generally, the issue of how well (or even in which sense), the Fourier Series in (1.1) or in (1.3) converges to the function F is a rather subtle one. *The main point of this problem is to conduct a numerical exploration of some aspects of this question.* In particular, consider the **partial sums:**

$$F_N(x) = c_0 + \sum_{n=1}^{N} (c_n \cos(n\,x) + s_n \sin(n\,x)), \tag{1.8}$$

where N is some natural number.

Important questions are then: **How well does F_N approximate the function F?** and **How big is the error**[1] **and how fast does it vanish as $N \to \infty$?**

Remark 1.1 *An important element in answering the questions above is how fast the Fourier coefficients vanish as $n \to \infty$. This is determined by how fast the* **power spectrum**

$$P_n = \frac{1}{2}|F_n| = \sqrt{c_n^2 + s_n^2} \tag{1.9}$$

vanishes as $n \to \infty$ **(assume F is real valued, so that (1.7) applies).**

P_n gives information on "how important" the n-th mode is in the Fourier Series. The name follows from the fact that in many physical situations one can interpret the square of the amplitude of the n-th Fourier coefficient, P_n^2, as the amount of energy in the n-th mode of the solution — e.g.: for the wave equation.

1.1.2 The problem to be done

This problem objective is to "experimentally" **study how Fourier series converge.** For this purpose you should use the following MatLab scripts

FouSerRedame.m fourierSC.m FSFun.m FSoption.m FSoptionP.m heatSln.m MakeButtonFSC.m

Put the scripts in a directory and start MatLab there. The help command will work as usual, in particular: help FouSerReadme gives a description of all the scripts. Each script has its own detailed description. The **script you need is** fourierSC. The others (except for heatSln) are helper scripts.

IMPORTANT:

- When you start the script fourierSC, it will ask you the questions:
  A. *Do you want to use the fancy (with buttons) or the plain interface?*
  B. *Up to how many terms in the Fourier series do you want to compute?*
  C. *For which values you want to plot?*

  About **B** and **C**: Calculations will be done (and the results shown) for the partial sums in (1.8), for the values N = 0, Nskip, 2*Nskip, . . . , Np. You will be asked to input Nskip and Np. About **A**: the fancy version does not work for some versions of MatLab.

- After you finish answering the questions, fourierSC will present you with a list of options for functions whose Fourier series it can compute: "user's choice", and pre-selected. **Check the scripts code to make sure you that understand exactly which functions you are dealing with.**

[1] Note also that there are many ways that the error can be measured: point-wise; least-squares; etc.

- The script FSFun.m is the one used to input the "user's choice" selection — whatever function you program there will be the one used when "user's choice" is selected. A trivial example is pre-programmed in FSFun.m, but **you should alter it,** and write there any function for which you want to investigate the Fourier series, **to go beyond** the preselected options, which I encourage you to do.
- The pre-selected options include smooth functions, as well as functions with various types of singular behaviors — discontinuities, corners and cusps. *The idea is to investigate how any particular "singular" behavior in the function is related to the convergence properties of its Fourier series.*

  A **cusp is a singularity** such as the one that $\sqrt{|x|}$ has at the origin. Other possibilities are $|x|^{\alpha}$, where $0 < \alpha < 1$. *Investigate the effect of singularities of this type on the convergence.*

  The most important singular behavior whose effect on the Fourier series you should elucidate is that of a discontinuity. How does it affect the convergence? How do the partial sums look like in this case? Is there any peculiar behavior you can observe?

  Odd an even functions are also provided in the pre-selections, so that you can see what effect symmetries of the function have on its Fourier series. Can you think of other symmetries?
- The script fourierSC makes lots of plots, which will be made one on top of the other. You *need to move the windows to see all the plots.* **These plots illustrate various aspects of how a Fourier series behaves, as follows** (this is the order in which the plots are done):
  - **Exact function whose Fourier series is being computed.**
  - **Sine Fourier coefficients s_n, as a function of n.**
  - **Cosine Fourier coefficients c_n, as a function of n.**
  - **Semi-log plot of the power spectrum $P_n = \sqrt{c_n^2 + s_n^2}$ as a function of n** — exponential decay yields a straight line in this kind of plot.
  - **Log-log plot of the power spectrum as a function of n** — algebraic decay yields a straight line in this kind of plot.
  - **Partial sums $F_N = F_N(x)$ — as in equation (1.8) — for** N = 0:Nskip:Np. All these plots will be shown in the same window, so you must look at them as they are done.
  - **Relative error in the approximation F_N, as a function of N** — shows the error in the partial sums in (1.8), as a function of N, for N = 1:Np.
  - **Semi-log plot of the relative error, as a function of N.**
  - **Log-log plot of the relative error, as a function of N.**

This is what you are expected to do: Use the script fourierSC and experiment with the various choices. Then report any "pattern" or peculiar behavior you observe in the way Fourier series converge. The plots are useful in figuring out how fast things converge (e.g. how fast do the Fourier coefficients vanish as $n \to \infty$). Look at the plots, look for patterns and trends. Make hypothesis as to what is happening and **test them by further experimentation** — use the script FSFun to produce functions where you can test your hypothesis. Write your conclusions in the answers. **Describe the evidence for your conclusions** — no proof is required, numerical evidence is enough, but **you must produce, and describe, the evidence!** Think of it in the same way that you would think in the situation of a lab experimenter trying to figure out what happens in some problem. **A few plots with your answer are fine, but please, just a few!**

Important: the point of this problem is not for you to run code and show pretty pictures of what you see. I want you to arrive at quantitative conclusions about how Fourier series converge, etc. The plots for the pre-programmed functions will show you patterns that you need to recognize, and test with other functions to identify what causes them. I **expect statements that sound like this:** "if a function has the following property ... (discontinuity, cusp, corner, whatever), then the power spectrum decays like ...; the error in F_N behaves like ...; etc."

Important: Anything smaller than about 10^{-14} is numerical error. Ignore it!

1.2 Answer: Computer Exercise in Fourier Series

There are very many properties that you could have discovered and given in answer to this problem. A few are listed in what follows.

An obvious first observation is that, **if F is even, then the sine Fourier coefficients s_n vanish,** while **if F is odd, then the cosine Fourier coefficients c_n vanish.** This observation is very easy to prove using the formulas in (1.4 – 1.6). Other symmetries in the function F have similar consequences on the coefficients; for example: **what does $F(\pi - x) = F(\pi + x)$ yield?**

A second fairly immediate observation should be that, in the examples where **F is smooth (infinitely many derivatives), just a few terms in the Fourier series give amazingly good approximations** — within 20 terms or so the error becomes less than 10^{-15}, at the limit of the numerical resolution in the MatLab script. On the other end, in the example where there is a **corner in F (F continuous but the first derivative has a discontinuity), the error goes down very slowly.**[2] Further, in the extreme cases where **F has discontinuities,** at the location of the discontinuity there is an **error that never seems to go down to zero!** — more on this below. Thus,

There is a direct correspondence between the degree of smoothness in F and how "good" the Fourier series is.

This observation can be understood by the following simple calculation. Assume that F in (1.2) has an integrable derivative. Then, using the periodicity, a simple integration by parts shows that (for $n \neq 0$):

$$F_n = \frac{1}{2in\pi} \int_{-\pi}^{\pi} \frac{dF}{dx}\, e^{-i\,n\,x}\, dx. \tag{1.10}$$

If F has two derivatives, then we can do this again, and so on. Note that, each time we do this, an extra factor of n appears in the denominator. What this means is that, the smoother F is (the more derivatives it has), the faster the Fourier coefficients decay as $n \to \infty$. Of course, this means that (1.1) converges faster. In particular, **if F has infinitely many derivatives, the Fourier coefficients decay faster than any power of n $\Rightarrow$** incredibly good convergence. Furthermore: **it can be proved that, if F is analytic, then its Fourier coefficients decay exponentially as $n \to \infty$.**

> *Go back and check these facts with the MatLab scripts! Many of the examples (i.e.: the Gaussians) have infinitely many derivatives. The semi-log and log-log plots of the power spectrum $P_n = \sqrt{c_n^2 + s_n^2}$ make these properties evident.*

On the other hand, consider the **example with the corner.** For this particular example we can calculate the Fourier coefficients exactly — since the function is piecewise linear **(do it)**. What you will find is that **the Fourier coefficients decay (as $n \to \infty$) like n^{-2}.** Again, you can verify this with MatLab: look at the semi-log plots of the power spectrum.[3] It can be shown that **this decay rate is a general feature for functions F that have corners (but are, otherwise, "well behaved").** For them the series in (1.1) converges roughly like the series $\sum \frac{1}{n^2}$ — that is: very slow convergence.

Another **example that you should look at is that of the "rounded hat".** The *almost corner this example has at the top is not quite a corner: a derivative exists there, but it behaves badly (very much the way $|x|^{1.5}$ behaves near $x = 0$.)* In this case **the Fourier coefficients decay like $n^{-2.5}$** — hence, a bit better than in the case of the hat, but not that much better.

Finally, the most interesting question: **What happens when F has a discontinuity?**
Consider the **sawtooth and square wave examples.**
In these cases the function F is either piecewise linear or piecewise constant, so that *one can compute exactly the Fourier coefficients F_n.* If you do so, you will find out that they **decay (as $n \to \infty$) like n^{-1}. This is a general**

[2] Furthermore: the plots show that the error occurs mostly near the corner.
[3] Note that the even indexed coefficients vanish here; thus look only at the decay of the nonzero coefficients.

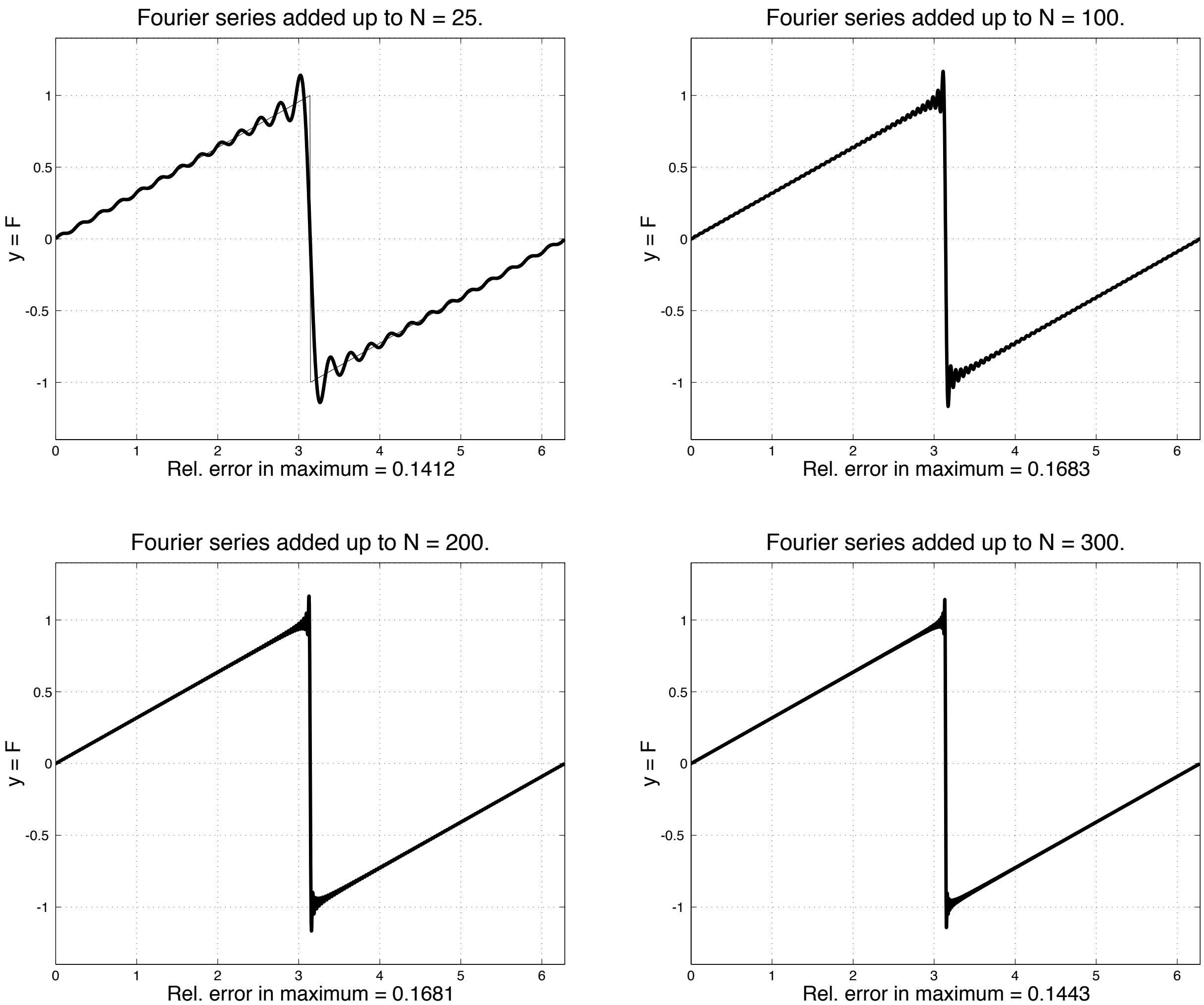

Figure 1.1: Computer Exercise in Fourier Series: Gibbs phenomenon. Convergence of the Fourier series for a sawtooth function. Notice the oscillations near (and overshot at) the position of the discontinuity.

property of functions with discontinuities and it is **very bad news!** As you know, the harmonic series $\sum(1/n)$ does not converge at all, while $\sum((-1)^n/n)$ has extremely poor convergence properties. But this is, precisely, the way that (1.1) converges in cases with discontinuities. In fact, the following happens with the partial sums F_N in (1.8) as $N \to \infty$:

> **Near the discontinuity a thin region with oscillations arises. As the number of terms in the series grows, this region becomes thinner, but the oscillations do not disappear; they just develop of a shorter wavelength. The amplitude of the oscillations does not vanish: There is an "overshot" at the discontinuity, which tends to a limit amplitude which is a constant fraction of the jump at the discontinuity.**

This is called the **"Gibbs" phenomenon** (see figures 1.1 and 1.2), a rather annoying[4] property of Fourier series. The

[4] For those that need to use Fourier Series to approximate functions that may not be too "nice".

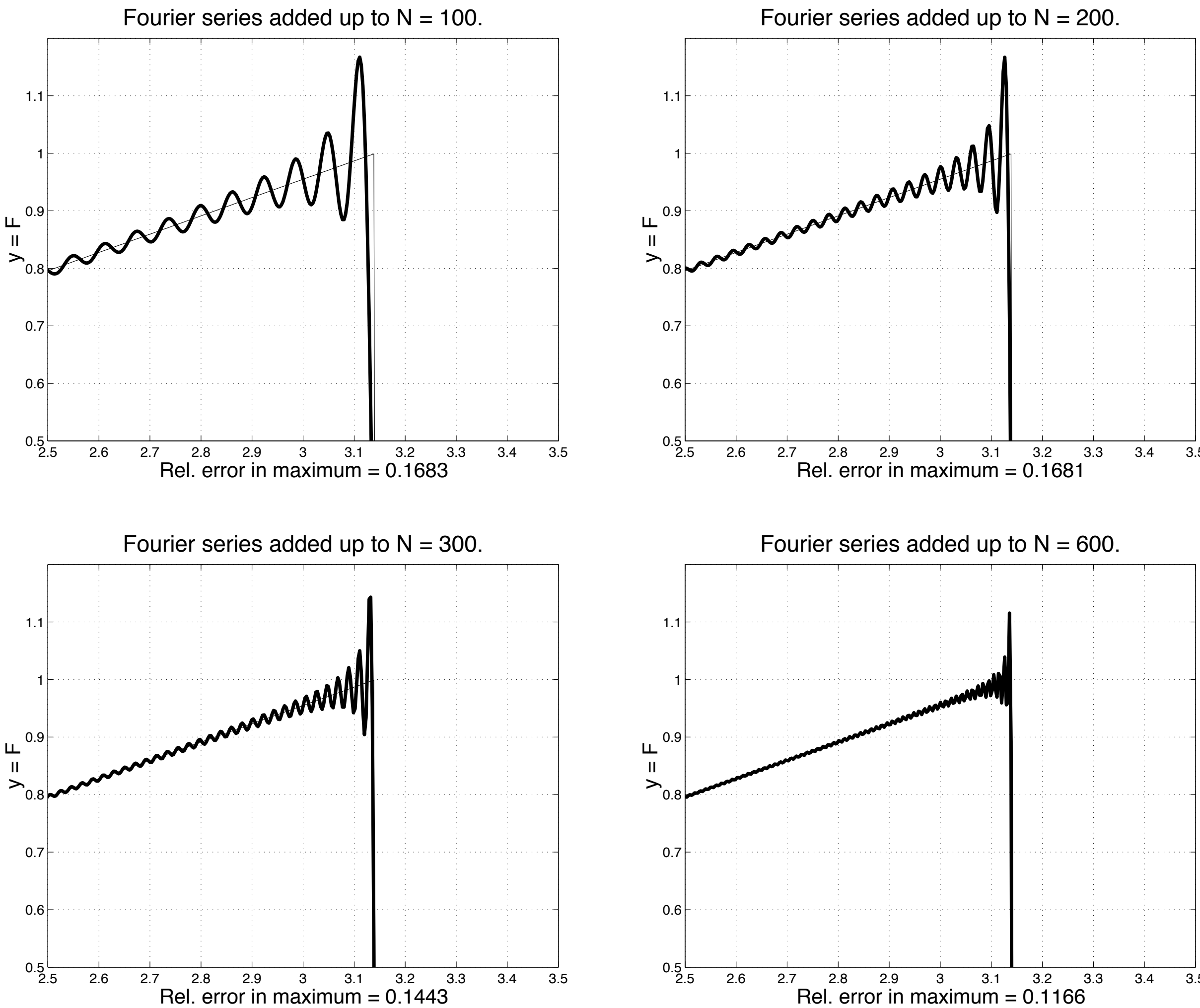

Figure 1.2: Computer Exercise in Fourier Series: Gibbs phenomenon. Detail near the discontinuity. As more and more terms are added, the overshot at the discontinuity does not go down, with very high frequencies occurring near it.

MatLab script gibbsFSE.m gives a demonstration of the details of this phenomenon.

2 N torsion coupled pendulums, and continuum limit as $N \to \infty$

2.1 Statement: N torsion coupled pendulums, and continuum limit as $N \to \infty$

Generalize the result of the problem "Two torsion coupled pendulums" to the case where there are $\boldsymbol{N}$ equal rods attached to the axle (equally spaced along it, so that they are a distance $\boldsymbol{\ell/(N+1)}$ from each other and from the axle's ends), each with a mass $\boldsymbol{M/N}$ at its end.

Let $\boldsymbol{x}$ be the length coordinate along the axle. Label the rods (starting from one end of the axle, at which we set $x = 0$) by the integers $n = 1, \dots, N$. Then the $\boldsymbol{n^{th}}$ rod corresponds to the position $\boldsymbol{x = x_n = \dfrac{n}{N+1}\,\ell}$ along the axis, and it is characterized by the angle $\boldsymbol{\theta_n = \theta_n(t)}$.

Consider now the limit $N \to \infty$. Look at solutions for which you can write $\boldsymbol{\theta_n(t) = \theta(x_n,\, t)}$, where $\boldsymbol{\theta = \theta(x,\, t)}$ is a "nice" function (i.e.: $\boldsymbol{\theta}$ has as many derivatives as you need, so that Taylor expansions are valid). **Derive a P.D.E.**[5] **for the function $\boldsymbol{\theta = \theta(x,\, t)}$.**

Hint 2.1 (For the continuum limit $N \to \infty$). (i) *The pendulum masses, M/N, scale with N so as to produce a constant mass per unit length (density) $\rho = M/\ell$ in the limit $N \to \infty$.* **(ii)** *The pendulum positions, $x_n = n\,\ell/(N+1)$, approach a continuous distribution as $N \to \infty$.* **(iii)** *See the hint in the problem "Pendulum with torsion" for the appropriate way in which the torsional force between any two neighboring rods scales with N.* **(iv)** *Note that, in the limit $N \to \infty$, angle differences such as $\theta_{n+1} - 2\,\theta_n + \theta_{n-1}$, can be approximated in terms of derivatives of θ with respect to x* (Taylor expand $\theta_{n\pm1}$ centered at x_n).

2.2 Answer: N torsion coupled pendulums, and continuum limit as $N \to \infty$

Let $\boldsymbol{\theta_n}$ (for $\boldsymbol{n = 1, \ldots, N}$) be the angles for the positions of the $\boldsymbol{N}$ masses. The tangential force due to gravity on each of the masses is

$$F_g = -\frac{1}{N}\,M\,g \sin\theta_n, \quad \text{where} \quad n = 1, \ldots, N. \tag{2.1}$$

For any two successive masses, there is also a torque whenever $\boldsymbol{\theta_n \neq \theta_{n+1}}$ — generated by the twist in the axle of magnitude $\theta_{n+1} - \theta_n$, over the segment of length $\ell/(N+1)$ connecting the two rods. Thus each of the masses experiences a force (equal in magnitude and opposite in sign)

$$F_T = \pm\,(N+1)\,\frac{\kappa}{\ell\,L}\,(\theta_{n+1} - \theta_n), \tag{2.2}$$

where the signs are such that these forces tend to make $\theta_n = \theta_{n+1}$.

Hence the angles satisfy the equations

$$\frac{1}{N}\,M\,L\,\frac{d^2\theta_1}{dt^2} = -\frac{1}{N}\,M\,g\,\sin\theta_1 + \frac{(N+1)\,\kappa}{\ell\,L}\,(\theta_2 - \theta_1), \tag{2.3}$$

$$\frac{1}{N}\,M\,L\,\frac{d^2\theta_n}{dt^2} = -\frac{1}{N}\,M\,g\,\sin\theta_n + \frac{(N+1)\,\kappa}{\ell\,L}\,(\theta_{n+1} - \theta_n) - \frac{(N+1)\,\kappa}{\ell\,L}\,(\theta_n - \theta_{n-1}), \tag{2.4}$$

for $\boldsymbol{n = 2, \ldots, N-1}$, and

$$\frac{1}{N}\,M\,L\,\frac{d^2\theta_N}{dt^2} = -\frac{1}{N}\,M\,g\,\sin\theta_N - \frac{(N+1)\,\kappa}{\ell\,L}(\theta_N - \theta_{N-1}). \tag{2.5}$$

Remark 2.1 *Check that the signs for the torsion forces selected in these equations are correct by taking the difference between the n^{th} and $(n+1)^{th}$ equation. Then you should see that the torsion forces are acting so as to make the angles equal.*

Remark 2.2 *The equations for the first and last angle differ, because they experience a torsion force from only one side.* **How would you modify these equations to account for having one (or both) ends of the axle fixed?**

[5] The equation you will obtain is known as the **Sine-Gordon equation.**

Continuum limit.

Now we consider the continuum limit, in which we let $N \to \infty$ and assume that the n^{th} angle can be written in the form:

$$\theta_n(t) = \theta(x_n,\, t), \tag{2.6}$$

where $\theta = \theta(x,\, t)$ is a "nice" function (with derivatives) and $x_n = \dfrac{n}{N+1}\,\ell$. In particular, note that:

$$\Delta x = x_{n+1} - x_n = \frac{\ell}{N+1}. \tag{2.7}$$

Take equation (2.4), and multiply it by N/ℓ. Thus we obtain

$$\rho\, L\, \frac{d^2\theta_n}{dt^2} = -\rho\, g\, \sin\theta_n + \frac{N\,(N+1)\,\kappa}{\ell^2\, L}\,(\theta_{n+1} - 2\,\theta_n + \theta_{n-1}),$$

where $\boldsymbol{\rho = M/\ell}$ is the mass density per unit length in the $\boldsymbol{N \to \infty}$ limit. Using equation (2.7), this can be written in the form:

$$\rho\, L\, \frac{d^2\theta_n}{dt^2} = -\rho\, g\, \sin\theta_n + \frac{N}{(N+1)}\,\frac{\kappa}{L}\,\frac{\theta_{n+1} - 2\,\theta_n + \theta_{n-1}}{(\Delta x)^2}. \tag{2.8}$$

Using equation (2.6) we see that — in the limit $N \to \infty$ (where $\Delta x \to 0$) — we have:

$$\frac{\theta_{n+1} - 2\,\theta_n + \theta_{n-1}}{(\Delta x)^2} \to \frac{\partial^2\theta}{\partial x^2}(x_n,\, t).$$

Thus, finally, we obtain (for the continuum limit) the P.D.E. (the **"Sine–Gordon" equation**)

$$\boldsymbol{\theta_{tt} - c^2\,\theta_{xx} = -\omega^2\,\sin\theta,} \tag{2.9}$$

where $\boldsymbol{\omega = \sqrt{\dfrac{g}{L}}}$ is the pendulum angular frequency, and $\boldsymbol{c = \sqrt{\dfrac{\kappa}{\rho\, L^2}}}$ is a wave propagation speed — **check that the dimensions are correct.**

Remark 2.3 What happens with equations (2.3) and (2.5) in the limit $N \to \infty$? *As above, multiply (2.3) by* $1/\ell$. *Then*

$$\frac{\rho\, L}{N}\,\frac{d^2\theta_1}{dt^2} = -\frac{\rho\, g}{N}\,\sin\theta_1 + \frac{(N+1)\,\kappa}{\ell^2\, L}\,(\theta_2 - \theta_1) = -\frac{\rho\, g}{N}\,\sin\theta_1 + \frac{\kappa}{\ell\, L}\,\frac{\theta_2 - \theta_1}{\Delta x}.$$

Thus, as $\boldsymbol{N \to \infty}$, *we obtain*

$$\boldsymbol{\theta_x(0,\, t) = 0,}$$

which is just the statement that there are no torsion forces at the $\boldsymbol{x = 0}$ *end, since the axle is free to rotate there. In the same fashion,*

$$\boldsymbol{\theta_x(\ell,\, t) = 0.}$$

How would these (boundary) conditions change for an axle fixed at one (or both) ends?

3 Normal matrices

3.1 Statement: Normal matrices

A (square) matrix is normal if it commutes with its adjoint. That is: [6] $$\boldsymbol{A} * \boldsymbol{A}^\dagger = \boldsymbol{A}^\dagger * \boldsymbol{A}. \qquad (3.1)$$
We know that **normal matrices have an orthonormal basis of eigenvectors.**

Note #1.

A square matrix $\boldsymbol{A}$ is normal if and only if it can be written in the form $$\boldsymbol{A} = \boldsymbol{U} * \boldsymbol{D} * \boldsymbol{U}^\dagger, \qquad (3.2)$$
where $\boldsymbol{D}$ is diagonal, and $\boldsymbol{U}$ is unitary — **unitary means $\boldsymbol{U}^\dagger = \boldsymbol{U}^{-1}$.**

Proof. (Here we will use that $(\boldsymbol{A} * \boldsymbol{B})^\dagger = \boldsymbol{B}^\dagger * \boldsymbol{A}^\dagger$).

(a) If (3.2) applies, $\boldsymbol{A} * \boldsymbol{A}^\dagger = (\boldsymbol{U} * \boldsymbol{D} * \boldsymbol{U}^\dagger) * (\boldsymbol{U} * \boldsymbol{D} * \boldsymbol{U}^\dagger)^\dagger = (\boldsymbol{U} * \boldsymbol{D} * \boldsymbol{U}^\dagger) * (\boldsymbol{U} * \boldsymbol{D}^\dagger * \boldsymbol{U}^\dagger) = \boldsymbol{U} * \boldsymbol{D} * \boldsymbol{D}^\dagger * \boldsymbol{U}^\dagger$. Similarly $\boldsymbol{A}^\dagger * \boldsymbol{A} = \boldsymbol{U} * \boldsymbol{D}^\dagger * \boldsymbol{D} * \boldsymbol{U}^\dagger$. But $\boldsymbol{D}$ is diagonal, so that $\boldsymbol{D} * \boldsymbol{D}^\dagger = \boldsymbol{D}^\dagger * \boldsymbol{D}$. Hence $\boldsymbol{A}$ is normal. Note that the diagonal elements of $\boldsymbol{D}$ are the eigenvalues of $\boldsymbol{A}$. ‡ ♣

‡ Why? Let $\vec{\boldsymbol{u}}_n$ be the n-th column of $\boldsymbol{U}$. Then $\boldsymbol{U}^\dagger * \boldsymbol{U} = \boldsymbol{I}$ = identity implies that $\boldsymbol{U}^\dagger \vec{\boldsymbol{u}}_n = \{\delta_{nj}\}$ = vector with all entries zero but a 1 on the n-th one. Hence $\boldsymbol{A}\,\vec{\boldsymbol{u}}_n = \boldsymbol{\lambda}_n \vec{\boldsymbol{u}}_n$, where λ_n is the n-th diagonal element in $\boldsymbol{D}$.

(b) Suppose now that $\boldsymbol{A}$ is normal, and let $\{\vec{\boldsymbol{u}}_n\}_{n=1}^N$ be an orthonormal basis of eigenvectors. Hence
$$\boldsymbol{A}\,\vec{\boldsymbol{u}}_n = \boldsymbol{\lambda}_n\,\vec{\boldsymbol{u}}_n \quad \text{and} \quad \vec{\boldsymbol{u}}_m^\dagger * \vec{\boldsymbol{u}}_n = \boldsymbol{\delta}_{mn}, \qquad (3.3)$$
where the $\boldsymbol{\lambda}_n$ are the eigenvalues (complex constants), and $\boldsymbol{\delta}_{nm}$ is the Kronecker delta. Then, for any vector
$$\vec{\boldsymbol{v}} = \sum (\vec{\boldsymbol{u}}_n^\dagger * \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}}_n \;\Rightarrow\; \boldsymbol{A}\vec{\boldsymbol{v}} = \sum \boldsymbol{\lambda}_n (\vec{\boldsymbol{u}}_n^\dagger * \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}}_n. \qquad (3.4)$$
Let $\boldsymbol{U}$ = matrix whose columns are the $\boldsymbol{u}_n$, and $\boldsymbol{D}$ = diagonal matrix with diagonal elements $\boldsymbol{\lambda}_n$. Then the second equality in (3.4) yields $\boldsymbol{A}\,\vec{\boldsymbol{v}} = \boldsymbol{U} * \boldsymbol{D} * \boldsymbol{U}^\dagger \vec{\boldsymbol{v}}$. Since $\vec{\boldsymbol{v}}$ is arbitrary, this means that (3.2) applies. Finally: $\boldsymbol{U}$ is unitary because of the second equality in (3.3). ♣

Note #2.

Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^\dagger = p(\boldsymbol{A})$, where $\boldsymbol{p}$ is a polynomial. Then $\boldsymbol{A}$ is normal. (3.5)

Proof. Obvious, since $\boldsymbol{A}$ and $\boldsymbol{p}(\boldsymbol{A})$ commute.

Your task in this problem. Show that **if $\boldsymbol{A}$ is normal, then $\boldsymbol{A}^\dagger = p(\boldsymbol{A})$ for some polynomial.** (3.6)

Hint. Consider a polynomial such that $p(\lambda_n) = \bar{\lambda}_n$ for the eigenvalues of A (such polynomials exist: see Lagrange polynomial interpolation).

3.2 Answer: Normal matrices

Note that, because of (3.2), $\boldsymbol{A}^m = \boldsymbol{U} * \boldsymbol{D}^m * \boldsymbol{U}^\dagger$ for any natural number m. Hence, if $\boldsymbol{p}$ is the polynomial in the hint, $\boldsymbol{p}(\boldsymbol{A}) = \boldsymbol{U} * \boldsymbol{p}(\boldsymbol{D}) * \boldsymbol{U}^\dagger = \boldsymbol{U} * \boldsymbol{D}^\dagger * \boldsymbol{U}^\dagger = \boldsymbol{A}^\dagger$. ♣

The answer to the next problem will be provided with the answers to problem set #8.

4 Experiments with a slinky

Statement: Experiments with a slinky

Consider a homogeneous cylindrical rod (made of an elastic material), subject to (small amplitude) longitudinal deformations. Let $\boldsymbol{x}$ be the length coordinate (measured along the axis of the cylinder) when the cylinder is in its relaxed position. Use x as a label for the mass elements in the cylinder. [7] For every mass element x, let $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$

[6] **We use $*$ to denote matrix multiplication, and † to denote the adjoint (transpose conjugate).**

[7] Since we are considering only longitudinal motions, points in a cross section move in unison and we need not label them separately.

be its position at time t, measured along the axis of the cylinder (note that $u = x$ corresponds to the cylinder at rest.) Then u describes the state of the cylinder at any time and obeys the **wave equation:**

$$u_{tt} - c^2\, u_{xx} = 0, \quad \text{where} \quad c = \sqrt{k/\rho}. \tag{4.1}$$

Here ρ is the density (mass per unit length) of the rod, and k characterizes the elastic response of the material: if we stretch the cylinder by an amount ΔL, then the elastic force is $k\dfrac{\Delta L}{L}$, where L is the length of the rod (note that k has the dimensions of a force, thus c is a speed).

Remark 4.1 *The basic assumption here is that the cylinder remains at all times within the regime of applicability of Hooke's law. This means that the deformations (given by $u_x - 1$) are small enough everywhere. In particular, this also implies that variations in the cross-section of of the cylinder can be ignored (e.g.: if volume is preserved, the cross section will be larger in regions under compression than in those under tension).*

In the derivation of equation (4.1) it is assumed that the elastic forces are dominant, so that other forces (e.g.: gravity) can be ignored. For a rod with a vertical orientation, such that the elastic forces are not dominant over gravity, equation (4.1) must be modified to:

$$u_{tt} - c^2\, u_{xx} = -g, \tag{4.2}$$

where g is the acceleration of gravity, and we assume that the vertical coordinate x increases upwards. In particular, **consider the case of a rod hanging vertically without any motion** (i.e.: $u = u(x)$, with no time dependence), and measure x from the bottom of the rod. Then:

$$u = 0 \quad \text{and} \quad u_x = 1 \;\text{ at }\; x = 0, \tag{4.3}$$

where the second equation follows because there is no force at the lower end (no section of the rod below that must be supported). Then the equation for $u = u(x)$, namely:

$$c^2\, u_{x\,x} = g,$$

can be integrated to yield:

$$\boldsymbol{u = \frac{g}{2c^2}\, x^2 + x.} \tag{4.4}$$

A particular **example where this should apply to is that of a slinky. One objective of this problem is for you to check how well a slinky obeys equation (4.4).**

Proceed as follows:

1. Get a slinky in good condition and draw a straight line along its edge, parallel to the slinky's axis. Draw the line so that, when it reaches one end (the "bottom" end), it does so **at the end** of the coil that makes the slinky — i.e.: no more coil beyond the mark.
2. Starting from the "bottom" end of the slinky, name the points at which each coil is marked by the line as $n = 0, 1, \dots$ Then (if w is the width of a coil) when the slinky is at rest, the position of the n^{th} point is given by

$$x_n = n\, w.$$

3. To find w, measure the total length of the slinky, and divide this by the number of coils. You can also easily measure the "density" ρ of the coil by weighting it and dividing the result by its length.
4. Hang the slinky in a vertical position [8] (with the bottom end down) and wait till it is at rest. Then measure the distance u_n of the n^{th} point from the point $n = 0$ at the bottom. One way to do this is to have a measuring tape on a wall right behind the hanging slinky.

[8] For example, staple it to the underside of a shelf by a wall.

5. Equation (4.4) predicts that
$$u_n = \frac{g}{2c^2}\, x_n^2 + x_n = \frac{g\rho}{2k}\, x_n^2 + x_n. \tag{4.5}$$

6. **The question is now:** How well does equation (4.5) match your measurements? Of course, you do not have k, but you will have several values of n. If (4.5) applies, then
$$u_{n+1} - u_n = \Delta u_n = \frac{g\,\rho}{k}\, w^2\, n + \frac{g\,\rho}{2\,k}\, w^2 + w.$$
Thus a plot of $\boldsymbol{u_{n+1} - u_n}$ versus $\boldsymbol{n}$ should give a straight line with slope $\boldsymbol{g\,\rho\,w^2/k}$. From this you can get $\boldsymbol{k}$, which is the hardest quantity to measure directly in this context.

Next suspend the slinky from one end and set it to vibrate (longitudinally). In this case, if we set the origin for the coordinate x at the top (where the slinky does not move), the governing equation will still be (4.2), but the boundary conditions are now:
$$u(0,\, t) = 0 \quad \text{and} \quad u_x(-L,\, t) = 1, \tag{4.6}$$
where $\boldsymbol{L}$ **is the length of the slinky in its rest state.** The first condition here follows because $x = 0$ corresponds to the clamped end at the top, while the second simply states that there are no elastic forces at the bottom end (same reason used when deriving the second condition in (4.3)). It is easy to see that these conditions (and the equation) are satisfied by the function
$$u = a \sin\left(\frac{\pi}{2\,L}\, x\right) \sin\left(\frac{c\,\pi}{2\,L}\, t\right) + \frac{g}{2\,c^2}\, x^2 + \left(1 + \frac{g\,L}{c^2}\right)\, x, \tag{4.7}$$
where a is an arbitrary constant. This solution corresponds to an oscillation with **period** $\boldsymbol{T = \dfrac{4\,L}{c}}$**.**

Now, **continue the experiment:**

7. In the situation described in the paragraph above, measure the period of the slinky — do not try to measure a single period, time several and then divide by the number of periods timed.
8. Compare the result of your measurement of the period with the one given by the formula for T above — from the prior steps you can obtain a value for c.
9. Discuss the results of your experiment.

THE END.