

Answers to P-Set # 06, 18.300 MIT (Spring 2022)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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The "computer exercise in Fourier Series" is part of the NEXT problem set. It is included here so you can start early with it. Include your answer with p.-set #7, NOT with p.-set #6.

1 Computer Exercise in Fourier Series

Statement: Computer Exercise in Fourier Series

1.0.1 Introduction: Fourier Series

Generally, a 2π -periodic function $F = F(x)$ can be expressed in terms of its **Fourier Series** where the n^{th} complex Fourier coefficient F_n is defined by

$$F(x) = \sum_{n=-\infty}^{\infty} F_n e^{i n x}, \quad (1.1)$$

$$F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-i n x} dx, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (1.2)$$

An alternative formulation, obtained upon using $e^{-i n x} = \cos(n x) - i \sin(n x)$, is given by:

$$F(x) = c_0 + \sum_{n=1}^{\infty} (c_n \cos(n x) + s_n \sin(n x)), \quad (1.3)$$

where: (i) $c_0 = F_0$ and $c_n = (F_n + F_{-n})$ are the cosine Fourier coefficients, and (ii) $s_n = i(F_n - F_{-n})$ are the sine Fourier coefficients. Thus

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx, \quad (1.4)$$

$$c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(nx) dx, \quad \text{for: } n = 1, 2, 3, \dots \quad (1.5)$$

$$s_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(nx) dx, \quad \text{for: } n = 1, 2, 3, \dots \quad (1.6)$$

If F is real valued, then F_{-n} is the complex conjugate of F_n , so that

$$c_n = 2 \operatorname{Re}(F_n) \quad \text{and} \quad s_n = -2 \operatorname{Im}(F_n) \quad \text{for } n > 0. \quad (1.7)$$

Generally, the issue of how well (or even in which sense), the Fourier Series in (1.1) or in (1.3) converges to the function F is a rather subtle one. *The main point of this problem is to conduct a numerical exploration of some aspects of this question.* In particular,

consider the **partial sums**:

$$F_N(x) = c_0 + \sum_{n=1}^N (c_n \cos(nx) + s_n \sin(nx)), \quad (1.8)$$

where N is some natural number.

Important questions are then: **How well does F_N approximate the function F ?** and **How big is the error¹ and how fast does it vanish as $N \rightarrow \infty$?**

Remark 1.1 *An important element in answering the questions above is how fast the Fourier coefficients vanish as $n \rightarrow \infty$. This is determined by how fast the*

power spectrum

$$P_n = \frac{1}{2} |F_n|^2 = \sqrt{c_n^2 + s_n^2} \quad (1.9)$$

vanishes as $n \rightarrow \infty$ (assume F is real valued, so that (1.7) applies).

P_n gives information on “how important” the n -th mode is in the Fourier Series. The name follows from the fact that in many physical situations one can interpret the square of the amplitude of the n -th Fourier coefficient, P_n^2 , as the amount of energy in the n -th mode of the solution — e.g.: for the wave equation.

1.0.2 The problem to be done

This problem objective is to “experimentally” **study how Fourier series converge**. For this purpose you should use the following MatLab scripts

FouSerRedame.m fourierSC.m FSFun.m FSoption.m FSoptionP.m heatSln.m MakeButtonFSC.m

Put the scripts in a directory and start MatLab there. The help command will work as usual, in particular: help FouSerReadme gives a description of all the scripts. Each script has its own detailed description. The **script you need is fourierSC**. The others (except for heatSln) are helper scripts.

IMPORTANT:

- When you start the script fourierSC, it will ask you the questions:
 - A.** *Do you want to use the fancy (with buttons) or the plain interface?*
 - B.** *Up to how many terms in the Fourier series do you want to compute?*
 - C.** *For which values you want to plot?*

About **B** and **C**: Calculations will be done (and the results shown) for the partial sums in (1.8), for the values $N = 0, N_{\text{skip}}, 2*N_{\text{skip}}, \dots, N_p$. You will be asked to input N_{skip} and N_p . About **A**: the fancy version does not work for some versions of MatLab.

- After you finish answering the questions, fourierSC will present you with a list of options for functions whose Fourier series it can compute: “user’s choice”, and pre-selected. **Check the scripts code to make sure you that understand exactly which functions you are dealing with.**

¹ Note also that there are many ways that the error can be measured: point-wise; least-squares; etc.

- The script FSFun.m is the one used to input the “user’s choice” selection — whatever function you program there will be the one used when “user’s choice” is selected. A trivial example is pre-programmed in FSFun.m, but **you should alter it**, and write there any function for which you want to investigate the Fourier series, **to go beyond** the preselected options, which I encourage you to do.
- The pre-selected options include smooth functions, as well as functions with various types of singular behaviors — discontinuities, corners and cusps. *The idea is to investigate how any particular “singular” behavior in the function is related to the convergence properties of its Fourier series.*

A **cusp is a singularity** such as the one that $\sqrt{|x|}$ has at the origin. Other possibilities are $|x|^\alpha$, where $0 < \alpha < 1$. *Investigate the effect of singularities of this type on the convergence.*

The most important singular behavior whose effect on the Fourier series you should elucidate is that of a discontinuity. How does it affect the convergence? How do the partial sums look like in this case? Is there any peculiar behavior you can observe?

Odd and even functions are also provided in the pre-selections, so that you can see what effect symmetries of the function have on its Fourier series. Can you think of other symmetries?

- The script fourierSC makes lots of plots, which will be made one on top of the other. You *need to move the windows to see all the plots*. **These plots illustrate various aspects of how a Fourier series behaves, as follows** (this is the order in which the plots are done):
 - **Exact function whose Fourier series is being computed.**
 - **Sine Fourier coefficients s_n , as a function of n .**
 - **Cosine Fourier coefficients c_n , as a function of n .**
 - **Semi-log plot of the power spectrum $P_n = \sqrt{c_n^2 + s_n^2}$ as a function of n** — exponential decay yields a straight line in this kind of plot.
 - **Log-log plot of the power spectrum as a function of n** — algebraic decay yields a straight line in this kind of plot.
 - **Partial sums $F_N = F_N(x)$ — as in equation (1.8) — for $N = 0:Nskip:Np$.** All these plots will be shown in the same window, so you must look at them as they are done.
 - **Relative error in the approximation F_N , as a function of N** — shows the error in the partial sums in (1.8), as a function of N , for $N = 1:Np$.
 - **Semi-log plot of the relative error, as a function of N .**
 - **Log-log plot of the relative error, as a function of N .**

This is what you are expected to do: Use the script fourierSC and experiment with the various choices. Then report any “pattern” or peculiar behavior you observe in the way Fourier series converge. The plots are useful in figuring out how fast things converge (e.g. how fast do the Fourier coefficients vanish as $n \rightarrow \infty$). Look at the plots, look for patterns and trends. Make hypothesis as to what is happening and **test them by further experimentation** — use the script FSFun to produce functions where you can test your hypothesis. Write your conclusions in the answers. **Describe the evidence for your conclusions** — no proof is required, numerical evidence is enough, but **you must produce, and describe, the evidence!** Think of it in the same way that you would think in the situation of a lab experimenter trying to figure out what happens in some problem. **A few plots with your answer are fine, but please, just a few!**

Important: the point of this problem is not for you to run code and show pretty pictures of what you see. I want you to arrive at quantitative conclusions about how Fourier series converge, etc. The plots for the pre-programmed functions will show you patterns that you need to recognize, and test with other functions to identify what causes them. I **expect statements that sound like this:** “if a function has the following property ... (discontinuity, cusp,

To be specific: in the problem here, imagine that a section of length $\Delta\ell$ of the axle has been twisted by an amount (angle) Ψ . Then, if \mathbf{T} is the torque generated by this twist, we have

$$T = \frac{\kappa \Psi}{\Delta\ell},$$

where κ is a constant that depends on the axle material and the area of its cross-section (assuming that the axle is an homogeneous cylinder). **What are the dimensions of κ ?**

This torque translates on a tangential force of magnitude $F = T/L$ on the mass M at the end of the rod R of length L . The sign of the force is such that it acts to undo the twist.

Finally, note that in this problem, when the rod is at an angle θ from its vertical reference position, a twist of magnitude θ will be generated on any side where the axle ends are tied.

2.2 Answer: Pendulum with torsion

It all boils down to figuring out what Newton's law $F = ma$ is for this case. The assumptions in the problem indicate that we only have to worry about tangential forces and accelerations, since there is no motion in the radial direction. Now we have:

1. The mass is given by M .
2. The tangential acceleration is given by $L \frac{d^2\theta}{dt^2}$.
3. The tangential force due to gravity is given by $F_g = -M g \sin \theta$.
4. If either end of the axle A is fixed, then an angle θ for the pendulum will be associated with a twist in the axle of magnitude θ over a length $\ell/2$. This will give rise to a restoring torque $T = -2\kappa\theta/\ell$ (at least for small angles), thus a force $F_T = -2\kappa\theta/\ell L$ on the pendulum mass.
5. It follows that **the equation of motion for the case of a single end of the axle fixed is:**

$$M L \frac{d^2\theta}{dt^2} = -M g \sin \theta - \frac{2\kappa}{\ell L} \theta. \quad (2.2)$$

6. Similarly, **the equation of motion for the case with both ends of the axle fixed is:**

$$M L \frac{d^2\theta}{dt^2} = -M g \sin \theta - \frac{4\kappa}{\ell L} \theta. \quad (2.3)$$

7. The constant κ has dimensions of torque \times length = force \times length². Thus

$$\text{dimensions}(\kappa) = \text{mass} \frac{\text{length}^3}{\text{time}^2}. \quad (2.4)$$

Remark 2.1 On the sign of the forces: when writing the forces, the signs must be selected to agree with the fact that all the forces pull the mass towards the $\theta = 0$ position.

3 Two torsion coupled pendulums

3.1 Statement: Two torsion coupled pendulums

A variation on the problem "Pendulum with torsion" is as follows:

1. The axle has two equal rods attached to it (equally spaced along the axle, so that they are a distance $\ell/3$ from each other and from the ends).
2. Each rod has a mass $\frac{1}{2} M$ at its end.
3. The axle is free to rotate without friction.
4. The axle has no twist when both rods are pointing down.

Write the equations of motion for this system. Note that the state of the system is described by two angles, and that when writing the equations of motion you have to consider the torsional forces that are generated when the two angles are not equal. Assume that the angle differences are small, so Hooke's law applies and the torsional forces are proportional to the angle difference.

3.2 Answer: Two torsion coupled pendulums

Let θ_1 and θ_2 be the two angles for the positions of the two masses. The tangential force due to gravity on each mass is then

$$F_g = -\frac{1}{2} M g \sin \theta_n, \quad \text{where } n = 1 \text{ or } n = 2. \quad (3.1)$$

There is also a torque when $\theta_1 \neq \theta_2$ — generated by the twist in the axle of magnitude $\theta_1 - \theta_2$, over the segment of length $\ell/3$ connecting the two rods. Thus each of the masses experiences a tangential force (equal in magnitude and opposite in sign)

$$F_T = \pm 3 \frac{\kappa}{\ell L} (\theta_1 - \theta_2), \quad (3.2)$$

where the signs are such that these forces tend to make $\theta_1 = \theta_2$.

Hence the angles satisfy the equations

$$\frac{1}{2} M L \frac{d^2 \theta_1}{dt^2} = -\frac{1}{2} M g \sin \theta_1 - \frac{3 \kappa}{\ell L} (\theta_1 - \theta_2), \quad (3.3)$$

and

$$\frac{1}{2} M L \frac{d^2 \theta_2}{dt^2} = -\frac{1}{2} M g \sin \theta_2 + \frac{3 \kappa}{\ell L} (\theta_1 - \theta_2). \quad (3.4)$$

Remark 3.1 *If you have doubts about the signs for the torsion forces selected in these equations, take their difference. Then you will get an equation for $\theta_1 - \theta_2$, where you can easily see that the torsion forces are acting so as to make $\theta_1 - \theta_2 = 0$.*

4 SGDP04. Simple waves and Riemann invariants

4.1 Statement: Simple waves and Riemann invariants

Consider the equations of isentropic Gas Dynamics where $p = p(\rho)$ is increasing and $c = \sqrt{\frac{dp}{d\rho}}$. In the notation of remark 4.1, here

$$\rho_t + (\rho u)_x = 0 \quad \text{and} \quad u_t + u u_x + \frac{c^2}{\rho} \rho_x = 0, \quad (4.1)$$

$$\vec{Y} = \begin{pmatrix} \rho \\ u \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}. \quad (4.2)$$

Consider now the simple waves² associated with

$$\lambda = u + c \quad \text{and} \quad \vec{R} = \begin{pmatrix} 1 \\ c/\rho \end{pmatrix}. \quad (4.3)$$

Show that these are exactly the same solutions that result from taking $u - h = \text{constant}$ in (5.22–5.24). See hint 4.2.

² Note that (4.6) has more than one solution.

Remark 4.1 Simple wave solution. Consider a system of equations where $\vec{Y} = \vec{Y}(x, t)$ is column N -vector valued, and \mathbf{A} is $N \times N$ -matrix valued. Let $\lambda = \lambda(\vec{Y})$ be a real eigenvalue of \mathbf{A} , with corresponding right-eigenvector \vec{R} ,

$$\vec{Y}_t + \mathbf{A}(\vec{Y}) \vec{Y}_x = \mathbf{0}, \quad (4.4)$$

$$\mathbf{A} \vec{R} = \lambda \vec{R}. \quad (4.5)$$

Let $\vec{U} = \vec{U}(\phi)$ be a solution of the ode

$$\frac{d\vec{U}}{d\phi} = \vec{R}(\vec{U}). \quad (4.6)$$

Then it can be shown that

$$\vec{Y} = \vec{U}(\phi(x, t)) \quad (4.7)$$

is a solution to (4.4) provided that ϕ solves the scalar equation

$$\phi_t + \lambda(\vec{U}(\phi)) \phi_x = 0. \quad (4.8)$$

Solutions of this type are called “simple waves”, because they “excite” a single eigendirection of \mathbf{A} . Note that the solutions to (4.8) generally exhibit wave-steepening (and develop infinite derivatives in finite time) if where we have used the chain rule, and $\nabla\lambda$ denotes the gradient of λ with respect to \vec{Y} . An eigenvalue that satisfies $(\nabla\lambda) \cdot \vec{R} \neq 0$ is called **genuinely nonlinear**, while one satisfying $(\nabla\lambda) \cdot \vec{R} \equiv 0$ is said to be **linearly degenerate**. ♣

$$\mathbf{0} \neq \frac{d\lambda}{d\phi} = (\nabla\lambda) \cdot \vec{R}, \quad (4.9)$$

Hint 4.2 Note: $u - h = \text{constant}$ trivially satisfies (5.24). Further, since the definition of h in (5.22) involves an arbitrary constant of integration, we can assume (without loss of generality) $u - h = 0$. That is $u = h(\rho)$. Then (5.23) becomes $(u + h)_t + (u + c)(u + h)_x = 0$. (4.10) But $u + h = 2h$. Hence, using the chain rule and dividing by $2 dh/d\rho = 2c/\rho \neq 0$, (5.23) yields $\rho_t + (u + c)\rho_x = 0$, (4.11) where $u + c = h(\rho) + c(\rho)$. ♣

4.2 Answer: Simple waves and Riemann invariants

From (4.3) and (4.6), using (5.22),

$$U = \begin{pmatrix} \phi \\ h(\phi) \end{pmatrix}. \quad (4.12)$$

Thus the simple wave solution is

$$\begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} \phi \\ h(\phi) \end{pmatrix} \quad \text{and} \quad \phi_t + (u + c)\phi_x = 0, \quad (4.13)$$

where $u + c = h(\phi) + c(\phi)$.

Obviously, $\phi = \rho$, so this is the same as (4.11). ♣

5 1-D isentropic Euler equations of Gas Dynamics (supplementary)

Equations that govern the behavior of a gas can be derived using the same *Conservation Equation* techniques used to derive equations for the examples of Traffic Flow, River Flows, Shallow Water Waves, Modulations of Dispersive Waves, etc. The conserved quantities in this case are the mass, the momentum and the energy. The resulting equations are the **Euler equations of Gas Dynamics**.

Under certain conditions, one can assume that the entropy is a constant throughout the flow. Then the equations can be simplified, with the elimination of the equation for the conservation of energy. The **one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics** are

$$\left. \begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \end{aligned} \right\} \quad (5.14)$$

where $\rho = \rho(x, t)$, $u = u(x, t)$, and $p = p(x, t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation here implements the conservation of mass, and the second the conservation of momentum. We also need an equation relating the fluxes of the conserved quantities with the conserved densities — the analogue of the

equation $q = Q(\rho)$ in Traffic Flow. In this case this is provided by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \quad (5.15)$$

For example, for an ideal gas $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

The system of equations given by (5.14) is known as the

**Conservation Form of the Isentropic Equations
of Gas Dynamics, in Eulerian Coordinates.**

Remark 5.2 Normally, the pressure is a function of both the density and some other thermodynamic variable, such as the temperature. But the isentropic assumption allows us to write the pressure as a function of the gas mass density only. ♣

Introduce now the function $c = c(x, t) > 0$ (**c is the sound speed**) by³

$$c = C(\rho), \quad \text{where } C(\rho) = \sqrt{\frac{dP}{d\rho}}(\rho). \quad (5.16)$$

An alternative form of the equations in (5.14) is then given by

$$\left. \begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + \frac{c^2}{\rho}\rho_x + u u_x &= 0. \end{aligned} \right\} \quad (5.17)$$

Multiplying the first equation here by c/ρ , and adding (or subtracting) it to the second, yields the **characteristic form** of the equations

$$0 = (u_t + (u + c)u_x) + \frac{c}{\rho}(\rho_t + (u + c)\rho_x), \quad (5.18)$$

$$0 = (u_t + (u - c)u_x) - \frac{c}{\rho}(\rho_t + (u - c)\rho_x). \quad (5.19)$$

Equivalently

$$0 = \frac{du}{dt} + \frac{c}{\rho} \frac{d\rho}{dt} \quad \text{along} \quad \frac{dx}{dt} = u + c, \quad (5.20)$$

$$0 = \frac{du}{dt} - \frac{c}{\rho} \frac{d\rho}{dt} \quad \text{along} \quad \frac{dx}{dt} = u - c. \quad (5.21)$$

These last two equations show that, if we **introduce a new variable** $h = h(\rho)$, defined by

$$\frac{dh}{d\rho} = \frac{C(\rho)}{\rho}, \quad (5.22)$$

then

$$u + h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u + c, \quad (5.23)$$

$$u - h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u - c. \quad (5.24)$$

The variable $\mathcal{R} = u + h$ is the **right Riemann invariant**, while $\mathcal{L} = u - h$ is the **left Riemann invariant**.

THE END.

³ For an ideal gas, $c = \sqrt{\gamma p/\rho}$. For dry air at one atmosphere and 15 degrees Celsius: $p = 1.013 \times 10^6$ dyn/cm², $\rho = 1.226 \times 10^{-3}$ g/cm³, and $\gamma = 1.401$. Hence $c = 340.2$ m/s — measured value is $c = 340.6$ m/s.