Answers to P-Set # 05, 18.300 MIT (Spring 2022)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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1 Traveling wave solutions and shocks (BuHe01)

1.1 Statement: Traveling wave solutions and shocks (BuHe01)

Imagine that someone tells you that the following equation is a model for traffic flow:

$$c_t + c\, c_x = \nu\, c_{xt}, \tag{1.1}$$

where $\boldsymbol{\nu > 0}$ is "small" [1] and c is the the wave velocity — related to the car density via $\boldsymbol{c = \frac{dQ}{d\rho}}$. *The objective of this problem is to ascertain if (1.1) makes sense as model for Traffic Flow.* To this end, answer the **two** questions below.

[1] Note that ν has dimensions of length, so small means compared with some appropriate length scale.

Question #1. *Does the model have acceptable traveling wave "shock" solutions* .... $\boldsymbol{c = F(z)}$, *where* .... $\boldsymbol{z = \frac{x - U\,t}{\nu}}$, *and* $\boldsymbol{U}$ *is a constant?*

Here *"acceptable" means the following*

1a. The function F has finite limits as $z \to \pm\infty$, i.e.: $c_L = \lim_{z \to -\infty} F(z)$ and $c_R = \lim_{z \to +\infty} F(z)$. Further: the derivatives of F vanish as $z \to \pm\infty$, and $c_L \neq c_R$.

*This means that, as $\nu \to 0$, the solution c becomes a discontinuity traveling at speed U, with $c = c_L$ for $x < U\,t$ and $c = c_R$ for $x > U\,t$. That is, a **shock wave**.*

1b. The solution satisfies the Rankine-Hugoniot jump conditions .... $\boldsymbol{U = \frac{[Q]}{[\rho]}}$, where ρ_L and ρ_R are related to c_L and c_R via $c_L = \frac{dQ}{d\rho}(\rho_L)$ and $c_R = \frac{dQ}{d\rho}(\rho_R)$.
Assume that $Q = Q(\rho)$ is a quadratic traffic flow function — see remark **1.1**.

1c. The solution satisfies the Entropy condition .... $\boldsymbol{c_L > U > c_R}$.

To answer this question:

A. Find all the solutions satisfying **1a**. **Get explicit formulas for F and U** in terms of c_L, c_R, and z.
B. Check if the solutions that you found satisfy **1b.**
C. Check if the solutions that you found satisfy **1c.**
D. Finally, given **A-C**: *Does, so far, the equation make sense as a model for traffic flow?*

Hints.

- *Find the ode F satisfies. Show it can be reduced to the form $F' = P(F)$, where P = second order polynomial.*
- *Write P in terms of its two zeroes, c_1 and c_2, and express all the constants (e.g.: U) in terms of c_1 and c_2.*
- *Solve now the equation, and relate c_1 and c_2 to c_L and c_R. You are now ready to proceed with* **A-D**.
- *Remember that, while the density ρ has to be non-negative, wave speeds can have any sign.*

Question #2. As a second model check, *study the small perturbations from a constant state $c = c_0$.* Let $c = c_0 + u$, where u is "infinitesimal". Write the equation for u and look for solutions of the form .... $u = e^{i\,k\,x + \lambda\,t}$, where $-\infty < k < \infty$, and λ is some function of k.
How do these solutions behave? Is this reasonable for a traffic flow model?

Remark 1.1 An attempt at "justifying" (1.1) goes as follows:
It is not unreasonable to assume that the drivers not only respond to the local traffic density, but its rate of change as well. A simple way to model this is to write

$$q = Q(\rho) - \nu\rho_t, \tag{1.2}$$

for the flow rate $q = q(x,\,t)$, where $\nu > 0$ is a constant characterizing the drivers response to the local rate of change in the density (the reason ν should be positive, is that the normal driver's reaction to a density increase should be to slow down).

Substituting (1.2) into the equation for conservation of cars, yields:

$$\rho_t + c(\rho)\,\rho_x = \nu\rho_{tx}\,, \tag{1.3}$$

where $c = \dfrac{dQ}{d\rho}$. When Q is a quadratic function of ρ, c is a linear function of ρ and (1.3) is equivalent to (1.1).

1.2 Answer: Traveling wave solutions and shocks (BuHe01)

Substituting $c = F(z)$, where $z = \dfrac{x - U\,t}{\nu}$, into the pde gives the ode

$$(F - U)\,F' = -U\,F'', \tag{1.4}$$

where the primes indicate differentiation with respect to z. This equation can be integrated once, to obtain

$$U\,F' = U\,F - \frac{1}{2}\,F^2 + \kappa, \tag{1.5}$$

where κ is a constant of integration. The right hand side in this equation is a quadratic function of F, with maximum value at $F = U$, where it reaches the value $\frac{1}{2}\,U^2 + \kappa$. Therefore, for κ large enough, the right hand side in (1.5) has two real zeros, and the equation can be written in the form

$$U\,F' = -\frac{1}{2}\,(F - c_1)\,(F - c_2), \tag{1.6}$$

where $\boldsymbol{c_1 \geq c_2}$ **are constants,**

$$\boldsymbol{U = \frac{1}{2}\,(c_1 + c_2),} \quad \text{and} \quad \boldsymbol{\kappa = -\frac{1}{2}\,c_1\,c_2.} \tag{1.7}$$

As long as $U \neq 0$, the appropriate (and explicit) solutions to (1.6) follow from the hyperbolic tangent — since $y = \tanh(s)$ satisfies $y\prime = 1 - y^2$. That is [2]

$$F = U + \frac{c_1 - c_2}{2}\,\tanh\left(\frac{c_1 - c_2}{4\,U}\,z\right). \tag{1.8}$$

Assume that $\boldsymbol{c_1 > c_2}$**,** since the case $c_1 = c_2$ is trivial. Then, with $\boldsymbol{c_L = \lim_{z \to -\infty} F(z)}$ and $\boldsymbol{c_R = \lim_{z \to +\infty} F(z)}$,

$$\text{If } \boldsymbol{U > 0}, \qquad \boldsymbol{c_L = c_2} \text{ and } \boldsymbol{c_R = c_1.} \text{ Thus } \boldsymbol{c_L < U < c_R}. \tag{1.9}$$

$$\text{If } \boldsymbol{U < 0}, \qquad \boldsymbol{c_L = c_1} \text{ and } \boldsymbol{c_R = c_2.} \text{ Thus } \boldsymbol{c_L > U > c_R}. \tag{1.10}$$

Thus the model has traveling solutions satisfying **1a** for all values of U, except $U = 0$ (note that equation (1.5) yields $F =$ constant for $U = 0$). However:

1. **Rankine-Hugoniot jump conditions: satisfied,** since the wave velocity is given as the average of the characteristic velocities, which is correct for a quadratic flow function — see remark **1.1**.
2. **Entropy condition: violated when** $\boldsymbol{U > 0}$**,** as shown by (1.9).

Thus, we must **conclude that (1.1) is NOT a good model for traffic flow.** But we have the derivation in remark **1.1**, or so it seems. However, note:

> It is, indeed reasonable to assume that the drivers respond to the rate of change of the density. But, why should they respond to ρ_t alone, as in (1.2)? **The rate of change of the density a driver sees is** $\boldsymbol{r = \rho_t + u\,\rho_x}$**, not** $\boldsymbol{\rho_t}$! The drivers should also respond to the local gradient of the density ρ_x, specially when it is large.
>
> Since $\rho_t \approx -c\,\rho_x$ (at least away from shocks), it follows that $r = \rho_t + u\,\rho_x \approx (u - c)\,\rho_x$. But $u \geq c$, so ρ_x and r have the same sign. Thus a model with $q = Q(\rho) - \nu\,\rho_x$ will behave properly [this model is actually used], but one based in (1.2) will not. In fact $r \approx -\frac{u-c}{c}\,\rho_t$, so that **(1.2) gives the wrong sign for the correction whenever positive wave speeds are involved!** (1.11)

Let us now consider solutions of the form $c = c_0 + u$, where c_0 is a constant and u is very small. Then

$$u_t + c_0\,u_x = \nu\,u_{xt}. \tag{1.12}$$

This has solutions of the form $u = e^{i\,k\,x + \lambda\,t}$, provided that

$$\lambda = \frac{-i\,c_0\,k}{1 - i\,\nu\,k} = \frac{\nu\,c_0\,k^2}{1 + \nu^2\,k^2} - i\,\frac{c_0\,k}{1 + \nu^2\,k^2}. \tag{1.13}$$

[2] Equation (1.8) shows what happens for $U = 0$. As $U \to 0$, the transition between c_1 and c_2 becomes sharper. In the limit F becomes a step function — which is actually consistent as a solution to (1.4) when $U = 0$, provided we write it using the conservation form.

If $c_0 > 0$, these solutions grow (exponentially), which means that any uniform traffic flow with $c_0 > 0$ is unstable — at least according to this model.[3] This is rather strange, since $c_0 > 0$ corresponds to light traffic. **Another indication that this is not a good model for traffic flow.**

2 SGDP01. The equations in Lagrangian coordinates

2.1 Statement: The equations in Lagrangian coordinates

As mentioned in §5, in the form given by (5.22), the equations are said to be written in **Eulerian (or laboratory) coordinates.** The equations can also be written in **Lagrangian (or particle following) coordinates**, where the "space" coordinate is a label for each gas particle, rather than a position in space. The "label" used is, in fact, the *mass to the left* of each particle, as defined below.

To transform the equations in (5.22) to Lagrangian coordinates, first *select some arbitrary (but fixed) fluid particle* P, and let $x = x_p(t)$ be *the position of the particle* P *at any time.* Thus

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u(x_p,\, t). \tag{2.1}$$

Next, introduce the change of variables

$$(x,\, t) \quad \longrightarrow \quad (\sigma,\, t) \tag{2.2}$$

where $\sigma = \sigma(x, t)$ is defined by

$$\sigma = \sigma(x, t) = \int_{x_p}^{x} \rho(\zeta,\, t)\, \mathrm{d}\zeta. \tag{2.3}$$

Note that

a. **The variable $\boldsymbol{\sigma}$ has dimensions of mass**. In fact: *$\boldsymbol{\sigma}$ is the amount of mass in the gas between $\boldsymbol{x_p}$ and $\boldsymbol{x}$, with $\boldsymbol{\sigma \geq 0}$ if $\boldsymbol{x \geq x_p}$, and $\boldsymbol{\sigma \leq 0}$ if $\boldsymbol{x \leq x_p}$.*

b. If there is a finite amount of gas, we can take $x_p \equiv -\infty$. In this case σ is the mass of the gas to the left of the point x, and it is always non-negative.

c. If there is an impermeable wall somewhere (beginning or end of a closed pipe containing the gas), we can use as x_p the position of this wall.

$$\left.\begin{array}{l}\textbf{For simplicity, in what follows we assume that } \boldsymbol{\rho > 0} \textbf{ everywhere.}\\ \textbf{That is, that there is no vacuum anywhere.}\end{array}\right\} \tag{2.4}$$

Your tasks in this problem are stated in items 1–6 below. (2.5)

1. Show that: *If $x = X(t)$ is a curve defined by $\sigma(X,\, t) = constant$, then $x = X(t)$ is a particle path. That is*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u(X,\, t). \tag{2.6}$$

Vice-versa, if $x = X(t)$ is a particle path, then $\sigma(X,\, t) = constant.$

Thus $\boldsymbol{\sigma}$ **is a Lagrangian Coordinate,** i.e.: constant when following a fixed mass point in the gas.

Hint. Plug $x = X(t)$ into (2.3), and take $\frac{\mathbf{d}}{\mathbf{d}t}$. Then use conservation of mass [first equation in (5.22)] and (2.1).

[3] Not surprising, given the point made in (1.11)!

2. **Show that:**

$$\frac{\partial \sigma}{\partial x} = \rho, \quad \text{and} \quad \frac{\partial \sigma}{\partial t} = -\rho\, u. \tag{2.7}$$

   Hint. The first equation is trivial. For the other use conservation of mass [first equation in (5.22)] and (2.1).

3. **Show that:** For any given time, the transformation $x \to \sigma$ is invertible. Give a formula for the inverse, $\boldsymbol{x = \chi(\sigma,\, t)}$. Assume that you know the specific volume $\mathbf{v = 1/\rho = v(\sigma,\, t)}$ in the Lagrangian coordinates.

   Hint. Use the result from item 2, to show that σ is a strictly increasing function of x — hence it has an inverse. Then compute $\boldsymbol{\chi_\sigma}$. This yields a formula that can be integrated to obtain χ. The result has the same "flavor" as (2.3) — that is, an integral involving v.

4. **Show that:** under the change of variables in (2.2), the (isentropic) Euler equations of Gas Dynamics in (5.22) take the form

$$\left.\begin{array}{rcl} \mathrm{v}_t - u_\sigma & = & 0, \\ u_t + p_\sigma & = & 0, \end{array}\right\} \tag{2.8}$$

   where $\mathbf{v = 1/\rho =}$ **specific volume**, and $\boldsymbol{p}$ **is a function**[4] **of v via (5.23).** Note that $\dfrac{\mathbf{d}\boldsymbol{p}}{\mathbf{dv}} = -\boldsymbol{\rho^2\, c^2}$, where $\boldsymbol{c}$ **is the sound speed** (defined in (5.24)).

   The system of equations given by (2.8) is known as the

   Conservation Form of the Isentropic Equations of Gas Dynamics, in Lagrangian Coordinates.

   Hint. First, use the equations in (2.7) to obtain expressions for the partial derivatives of any function f in the Eulerian frame, in terms of partial derivatives in the Lagrangian fame. Then use these expressions to re-write the equations in the Lagrangian frame. Note that the computations are a little simpler if the form of the equations in (5.25) is used — where, in addition, the substitution $c^2\, \rho_x = p_x$ is made in the second equation.

   Remark 2.1 *Notice that the derivation of the system of equations in (2.8), as sketched by the hint above, involves manipulations with derivatives that are not valid when the solution is not smooth (i.e., shocks are present). Nevertheless, the system in conservation form in (2.8) is valid, even when shocks are present:*

   First: $\displaystyle\int_{\sigma_1}^{\sigma_2} \mathrm{v}\, \mathrm{d}\sigma$ is the distance between the particles with Lagrangian coordinates σ_1 and σ_2. Hence the integral of the first equation in (2.8) states that this distance evolves according to the difference in velocity between these two gas particles. It follows that *the first equation in (2.8) expresses the "conservation of distance", and it is valid even for solutions that are not smooth.*

   Second: $\displaystyle\int_{\sigma_1}^{\sigma_2} u\, \mathrm{d}\sigma$ is the total momentum of the gas between the particles with Lagrangian coordinates σ_1 and σ_2. As above, it follows that *the second equation in (2.8) expresses the "conservation of momentum", and it is valid even for solutions that are not smooth.* ♣

5. Using the first equation in (2.8) [*"conservation of distance"*], and the formula for $\chi = \chi(\sigma,\, t)$ that you derived in item 3, **obtain expressions for** $\boldsymbol{\chi_\sigma}$ **and** $\boldsymbol{\chi_t}$ **in terms of v and** $\boldsymbol{u}$**.** These expressions are the analog, in Lagrangian coordinates, of (2.7) in Eulerian coordinates.

   Hint. Note that x_p is the position (in Eulerian coordinates) of the particle with $\sigma = 0$ — hence $\dot{\boldsymbol{x}}_{\boldsymbol{p}}$ is the velocity u at $\sigma = 0$ (in Lagrangian coordinates).

[4] That is: $p = p(\mathrm{v})$. For an ideal gas $p = \kappa\, \mathrm{v}^{-\gamma}$.

6. **Write the system in (2.8) in characteristic form.** That is: the analog of (5.26–5.32).

   Hint. Use $\frac{dp}{dv} = -\rho^2 c^2$ to write (2.8) as a system of equations in v and u. Then consider combinations of the two equations of the form *(second equation) + α (first equation)*, and select α to obtain the characteristic equations (all the derivatives are along a single direction in space-time).

2.2 Answer: The equations in Lagrangian coordinates

The answers in the items below correspond to the items with questions in the problem statement.

1. Substituting $x = X(t)$ into equation (2.3), and taking a derivative with respect to time yields

$$\begin{aligned}\frac{d\sigma(X,\,t)}{dt} &= \rho(X,\,t)\frac{dX}{dt} - \rho(x_p,\,t)\frac{dx_p}{dt} + \int_{x_p}^{X}\rho_t(\zeta,\,t)\,d\zeta \\ &= \rho(X,\,t)\frac{dX}{dt} - \rho(x_p,\,t)\frac{dx_p}{dt} - (\rho\,u)\Big|_{x_p}^{X} \\ &= \rho(X,\,t)\left(\frac{dX}{dt} - u(X,\,t)\right),\end{aligned} \tag{2.9}$$

   where we used the equation for the conservation of mass [first equation in (5.22)] to obtain the second line, and (2.1) to obtain the third line. Hence (2.6) applies if and only if $\sigma(X,\,t) =$ constant.

2. That $\sigma_x = \rho$ follows trivially from (2.3). On the other hand, the partial derivative with respect to time of (2.3) yields

$$\begin{aligned}\sigma_t(x,\,t) &= \int_{x_p}^{x}\rho_t(\zeta,\,t)\,d\zeta - \rho(x_p,\,t)\frac{dx_p}{dt} \\ &= -(\rho\,u)\Big|_{x_p}^{x} - \rho(x_p,\,t)\frac{dx_p}{dt} \;=\; -\rho(x,\,t)u(x,\,t),\end{aligned} \tag{2.10}$$

   where we used the equation for the conservation of mass [first equation in (5.22)] to obtain the second line, and (2.1) to obtain the third line. This finishes the proof of (2.7).

3. Since $\sigma_x = \rho > 0$, as follows from item 2 and (2.4), for any fixed time σ *is a strictly increasing function of* x. Hence **it has an inverse $x = \chi(\sigma,\,t)$,** which satisfies

$$x = \chi\left(\sigma(x,\,t),\,t\right). \tag{2.11}$$

   Taking the partial derivative of this expression with respect to x, and using the equation on the left in (2.7), yields

$$1 = \chi_\sigma\left(\sigma(x,\,t),\,t\right)\sigma_x(x,\,t) = \chi_\sigma\left(\sigma(x,\,t),\,t\right)\rho(x,\,t). \tag{2.12}$$

   Hence $\boldsymbol{\chi_\sigma = \mathrm{v}}$, which can be integrated to yield

$$\chi = \chi(\sigma,\,t) = x_p(t) + \int_0^{\sigma}\mathrm{v}(s,\,t)\,ds, \tag{2.13}$$

   where the constant of integration follows because we know that $x = x_p$ corresponds to $\sigma = 0$.

   Note: When $x_p \equiv -\infty$ (see item b in the problem statement), we must take:

$$\chi = \chi(\sigma,\,t) = \int_{\sigma_0}^{\sigma}\mathrm{v}(s,\,t)\,ds, \tag{2.14}$$

   where $\sigma_0 = \displaystyle\int_{-\infty}^{0}\rho(\zeta,\,t)\,d\zeta > 0$.

   Why? Because, for there to be a finite amount of gas, ρ must vanish as $x \to -\infty$ (corresponds to $\sigma \to 0$). Hence v becomes singular as $\sigma \to 0$ (in fact, not integrable), and a formula like (2.13) cannot be used.

4. From (2.7) we see that, under the change of variables in (2.2), partial derivatives behave as follows

$$(f_t)_{\mathsf{Eulerian}} = (f_t - \rho\, u\, f_\sigma)_{\mathsf{Lagrangian}} \quad \text{and} \quad (f_x)_{\mathsf{Eulerian}} = (\rho\, f_\sigma)_{\mathsf{Lagrangian}}\,. \tag{2.15}$$

Hence, the first equation in (5.25) yields, in Lagrangian coordinates,

$$0 = \rho_t - \rho\, u\, \rho_\sigma + u\, \rho\, \rho_\sigma + \rho^2\, u_\sigma = \rho_t + \rho^2\, u_\sigma == -\rho^2 \left(-\frac{1}{\rho^2}\, \rho_t - u_\sigma\right).$$

This is equivalent to the first equation in (2.8). On the other hand, the second equation in (5.25) yields the second equation in (2.8)

$$0 = u_t - \rho\, u\, u_\sigma + p_\sigma + u\, \rho\, u_\sigma = u_t + p_\sigma,$$

upon use of $p_x = c^2\, \rho_x$.

5. From (2.13), and the first equation in (5.29), it follows that

$$\chi_t = u \qquad \text{and} \qquad \chi_\sigma = \mathrm{v}, \tag{2.16}$$

where we have used that x_p is the position (in Eulerian coordinates) of the particle with $\sigma = 0$ — hence

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u\big|_{\sigma=0}.$$

6. Using that $\dfrac{dp}{d\mathrm{v}} = -\rho^2\, c^2$, we can write the system in (2.8) in the form

$$\left.\begin{array}{rcrcl} \mathrm{v}_t & - & u_\sigma & = & 0, \\ u_t & - & \rho^2\, c^2\, \mathrm{v}_\sigma & = & 0. \end{array}\right\} \tag{2.17}$$

Multiplying the first equation here by ρc, and adding (or subtracting) it to the second, yields the **characteristic form** of the equations

$$0 = (u_t + \rho\, c\, u_\sigma) - \rho\, c\, (\mathrm{v}_t + \rho\, c\, \mathrm{v}_\sigma)\,, \tag{2.18}$$

$$0 = (u_t - \rho\, c\, u_\sigma) + \rho\, c\, (\mathrm{v}_t - \rho\, c\, \mathrm{v}_\sigma)\,. \tag{2.19}$$

Equivalently

$$0 = \frac{\mathrm{d}u}{\mathrm{d}t} - \rho\, c\, \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}t} \qquad \text{along} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \rho\, c, \tag{2.20}$$

$$0 = \frac{\mathrm{d}u}{\mathrm{d}t} + \rho\, c\, \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}t} \qquad \text{along} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\rho\, c. \tag{2.21}$$

In terms of $\boldsymbol{h}$, as defined in (5.30), these two equations take the form

$$u + h \quad \text{is constant along the characteristics} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \rho\, c, \tag{2.22}$$

$$u - h \quad \text{is constant along the characteristics} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\rho\, c, \tag{2.23}$$

where, as before, $\boldsymbol{u \pm h}$ are the **Riemann invariants**, and we have used that $\boldsymbol{\dfrac{\mathrm{d}h}{\mathrm{d}\mathrm{v}} = -\rho\, c}$.

3 SGDP02. Simple initial value problem

3.1 Statement: Simple initial value problem

Solve the equations[5] in (5.22), with $\boldsymbol{p = \frac{1}{2}\,\rho^2}$, with the initial conditions

$$u \equiv -\frac{1}{3} \quad \text{and} \quad \rho \equiv \frac{1}{9} \quad \text{for} \quad x < 0, \tag{3.1}$$

$$u \equiv \frac{1}{3} \quad \text{and} \quad \rho \equiv \frac{4}{9} \quad \text{for} \quad x > 0. \tag{3.2}$$

Display explicit formulas for u and ρ as functions of $(x,\, t)$, for all $-\infty < x < \infty$ and $t > 0$.

Hint. Write the equations, and initial values, using the Riemann invariants as dependent variables — see (5.30–5.32). Then convert back to $\boldsymbol{u}$ and $\boldsymbol{\rho}$.

Remark 3.1 *The form of the pressure here corresponds to a polytropic gas with $\gamma = 2$. Alternatively, the equations correspond to the shallow water system.* ♣

3.2 Answer: Simple initial value problem

From (5.24) and (5.30) it follows that

$$c = \sqrt{\rho} \quad \text{and} \quad h = 2\,c. \tag{3.3}$$

Hence the initial data can be written in the form

$$\mathcal{L} \equiv -1 \quad \text{and} \quad \mathcal{R} \equiv \frac{1}{3} \quad \text{for} \quad x < 0, \tag{3.4}$$

$$\mathcal{L} \equiv -1 \quad \text{and} \quad \mathcal{R} \equiv \frac{5}{3} \quad \text{for} \quad x > 0. \tag{3.5}$$

Thus, from (5.32),

$$\mathcal{L} \equiv -1 \quad \text{for all} \quad -\infty < x < \infty \quad \text{and} \quad t > 0. \tag{3.6}$$

On the other hand, we can write equation (5.31) in the form

$$\mathcal{R}_t + (u + c)\,\mathcal{R}_x = 0. \tag{3.7}$$

We can also write

$$u = \frac{1}{2}\,(\mathcal{R} + \mathcal{L}) = \frac{1}{2}\,(\mathcal{R} - 1) \quad \text{and} \quad c = \frac{1}{2}\,h = \frac{1}{4}\,(\mathcal{R} - \mathcal{L}) = \frac{1}{4}\,(\mathcal{R} + 1)\,. \tag{3.8}$$

Thus (3.7) is the same as

$$\mathcal{R}_t + \left(\frac{3}{4}\,\mathcal{R} - \frac{1}{4}\right)\,\mathcal{R}_x = 0. \tag{3.9}$$

Given the initial data in (3.4–3.5), this last equation has the solution (for $\boldsymbol{t > 0}$)

$$\boldsymbol{\mathcal{R}} = \begin{cases} \frac{1}{3} & \text{for} \quad \boldsymbol{x \leq 0}, \\ \frac{4\,x}{3\,t} + \frac{1}{3} & \text{for} \quad \boldsymbol{0 < x < t}, \\ \frac{5}{3} & \text{for} \quad \boldsymbol{t \leq x}, \end{cases} \tag{3.10}$$

where the middle region is an *expansion fan*, generated by the discontinuity at the origin in the initial data for $\mathcal{R}$. That is, note that the characteristic speed for (3.9) satisfies

$$\frac{3}{4}\,\mathcal{R} - \frac{1}{4} = \frac{x}{t} \quad \text{for} \quad 0 < x < t. \tag{3.11}$$

[5] Note that here we use a non-dimensional formulation.

From (3.6) and (3.10), it follows that

$$\boldsymbol{u} = \begin{cases} -\frac{1}{3} & \text{for} \quad \boldsymbol{x \le 0,} \\ \frac{2\,x}{3\,t} - \frac{1}{3} & \text{for} \quad \boldsymbol{0 < x < t,} \\ \frac{1}{3} & \text{for} \quad \boldsymbol{t \le x,} \end{cases} \quad \text{and} \quad \boldsymbol{\rho} = \begin{cases} \frac{1}{9} & \text{for} \quad \boldsymbol{x \le 0,} \\ \frac{1}{9}\left(\frac{x}{t} + 1\right)^2 & \text{for} \quad \boldsymbol{0 < x < t,} \\ \frac{4}{9} & \text{for} \quad \boldsymbol{t \le x.} \end{cases} \tag{3.12}$$

Finally, we point out that, if in (3.1–3.2) the values for $x < 0$ and $x > 0$ are exchanged, then the solution cannot be obtained as simply as above. The reason is that then in equation (3.9) the characteristics cross, and a continuous solution is not possible. Then the manipulations leading to the Riemann invariant form fail, since these assume differentiability. In fact, neither $\mathcal{L}$, nor $\mathcal{R}$ stay constant when the corresponding characteristics cross a shock wave.

4 Solitary wave for the KdV equation

4.1 Statement: Solitary wave for the KdV equation

4.1.1 Introduction

By introducing a diffusive term of the form $\nu\rho_{xx}$ (with ν small and positive) into the equation for traffic flow, one can resolve the structure of shocks as **traveling waves.** That is, the equation

$$\rho_t + c(\rho)\,\rho_x = \nu\,\rho_{xx}, \quad \text{where} \quad \frac{dc}{d\rho} \neq 0, \tag{4.1}$$

has smooth traveling wave solutions, that become discontinuous shock transitions as $\nu \downarrow 0$. To be precise, equation (4.1) has solutions of the form

$$\rho = f\left(\frac{x - V\,t - x_0}{\nu}\right), \tag{4.2}$$

where V and x_0 are constants, and $f = f(\zeta)$ is a smooth function with the properties:

A. $f(\zeta) \to \rho_0$ as $\zeta \to -\infty$.

B. $f(\zeta) \to \rho_1$ as $\zeta \to \infty$.

Thus, as $\nu \downarrow 0$, the solution above in (4.2) becomes a discontinuity (shock), traveling along the line $x = x_0 + V\,t$, and connecting the state ρ_0 behind with the state ρ_1 ahead. For $\nu > 0$ small, but finite, the discontinuity is resolved by this solution into a smooth transition (over a length scale proportional to ν) connecting the two sides of the shock jump. Furthermore: the **Entropy** and **Rankine Hugoniot** jump conditions also *follow from these solutions,* since a function f with the properties above exists if and only if

$$c(\rho_0) > c(\rho_1) \quad \text{and} \quad V = \frac{[\,q\,]}{[\,\rho\,]}. \tag{4.3}$$

In the particular case when c is a linear function of ρ (quadratic flow function $q = q(\rho)$), it is easy to see that equation (4.1) reduces to the **Burgers equation** for the characteristic speed c. That is:

$$c_t + cc_x = \nu\,c_{xx}. \tag{4.4}$$

This follows upon multiplying (4.1) by $\frac{dc}{d\rho}$ (a constant in this case) and using the chain rule.

The Burgers equation has smooth traveling waves $c = f\left(\dfrac{x - V\,t - x_0}{\nu}\right)$, that can be written explicitly in terms of elementary functions:

$$c = \frac{c_0 + c_1}{2} - \frac{c_0 - c_1}{2}\tanh\left(\frac{c_0 - c_1}{4}\,\zeta\right), \quad \text{where } \zeta = \frac{x - V\,t - x_0}{\nu} \text{ and } c_0 > c_1. \tag{4.5}$$

These connect the states c_0 as $x \to -\infty$ with c_1 as $x \to \infty$. The speed V is given by the appropriate shock relation $V = \dfrac{1}{2}(c_0 + c_1)$, and x_0 is arbitrary.

There are many conservative processes in nature where (at leading order) a nonlinear kinematic first order equation applies (i.e.: the same equation as in traffic flow, with some flow function $q = q(\rho)$, that depends on the details of the processes involved). In all these cases the leading order equations lead to wave steepening and breaking, that generally is stopped by the presence of physical processes that become important only when the gradients become large. However, **it is not generally true** that these higher order effects are dominated by dissipation — many other possibilities can occur. Below we consider one such alternative situation, which happens to be quite common

Remark 4.1 *The point made in the prior paragraph is very important, since shocks (as a resolution of the wave breaking caused by the kinematic wave steepening) are the answer* **only** *when the higher order effects are of a dissipative nature. One should be very careful about not introducing shocks into mathematical models of physical processes just because the models exhibit wave distortion and breaking. Many other behaviors are possible, some rather poorly understood. The answer to a mathematical breakdown in a model is almost never to be found purely by mathematical arguments; a careful look at the physical processes the equations attempt to model is a must when this happens.*

A possible, and quite frequent, alternative to dissipation is dispersion: namely, the wave speed is a non-trivial function of the wave number. The simplest instance of dispersion introduces a term proportional to the third space derivative of the solution in the equations. When coupled with the simplest kind of nonlinearity (quadratic), this gives rise to the **Korteweg-de Vries (KdV) equation.** The nondimensional form of the KdV equation is:

$$\boldsymbol{u_t + u\,u_x = \epsilon^2 u_{xxx},} \tag{4.6}$$

where $\boldsymbol{\epsilon > 0}$ is a parameter expressing the ratio of dispersion (different wavelengths moving at different speeds) to nonlinearity.

Remark 4.2 *The KdV equation describes (for example) the propagation of long, small (but finite) amplitude, waves in shallow water channels. The KdV equation is a "canonical" equation that arises in very many dispersive situations. That this should be so is relatively simple to see:*
First. *Consider a wave situation where the wave phase speed depends on the wave number in a nontrivial way:* $c_p = c_p(k)$*. Clearly,* c_p *should be an even function of* k*, since both* k *and* $-k$ *correspond to the same wavelength. Thus, for long waves (*k *small) one should be able to expand* c_p *in the form* $c_p = \alpha + \beta\,k^2 + \dots$ *(where* α *and* β *are constants). Furthermore, by changing coordinates into a moving frame, we can always assume* $\alpha = 0$.
Second. $c_p \approx \beta\,k^2$ *corresponds to a relationship between the wave number and the wave frequency of the form* $\omega = k\,c_p(k) \approx \beta\,k^3$*, which corresponds to the equation* $u_t = \beta u_{xxx}$*. Thus we see how a third order derivative arises as the simplest example of dispersive behavior for long waves.*

4.1.2 The problem to do

Show that: **for the KdV equation above in (4.6), no shocks are possible.** To be precise, consider the non-trivial (i.e. $u \neq$ constant) traveling wave solutions of (4.6):

$$u = F(\zeta), \quad \text{where} \quad \zeta = \frac{x - V\,t - x_0}{\epsilon}, \tag{4.7}$$

$F = F(\zeta)$ is some smooth function, V is a constant (the wave speed), and x_0 is some arbitrary constant. Study all the solutions of this form such that (for some constants u_0 and u_1)

(a) $F(\zeta) \to u_0$ as $\zeta \to -\infty$, $F(\zeta) \to u_1$ as $\zeta \to \infty$, and F is not identically constant.

(b) The first and all the higher order derivatives of $F = F(\zeta)$ vanish as $\zeta \to \pm\infty$.

Then

$$\boxed{\textbf{Show that the conditions: } V < u_0 = u_1\textbf{, must apply.}} \tag{4.8}$$

These solutions cannot represent shock transitions, since the states at $\pm\infty$ are equal (therefore, no jump occurs). In fact, it is possible to write these solutions explicitly in terms of elementary functions (hyperbolic secant), but finding this explicit form is **optional.**

Remark 4.3 *The traveling waves described above (that you are supposed to analyze) are called* **solitary waves***, because they consist of a single isolated disturbance that vanishes very quickly as $x \to \pm\infty$. Such waves are easy to see in shallow water situations, where disturbances of a wavelength much bigger than the depth are generated. For example, in early summer many lakes develop a thin layer (a few feet thick) of warm water over the colder (heavier) rest of the water. Waves on this top layer, with wavelengths much longer than its thickness, obey an equation similar to the KdV equation, also supporting solitary waves. These are easy to see (when they are generated by boats) as traveling bumps on the surface of the lake — these are "one-dimensional" bumps, that is: they decay only in the direction of propagation; the surface elevation is (basically) independent of the direction normal to the propagation direction (thus they look like "rolls", moving on the lake surface).*
Note that the solitary waves for (4.6) are actually "dips", not bumps. *In order to obtain "bumps", the sign of the dispersive term in (4.6) must be reversed. Namely, one must consider the equation*

$$v_t + v\, v_x = -\epsilon^2 v_{xxx}.$$

The traveling waves for this equation, and those for (4.6), are related. In fact: $u = f(x - V\,t)$ is a traveling wave for (4.6), if and only if $v = -f(x + V\,t)$ is a traveling wave for the equation above.

Hint 4.1 *First: substitute (4.7) into the KdV equation (4.6). This will yield a third order ODE for the function $F = F(\zeta)$. Notice that the parameter ϵ is used in (4.7) to scale the wavelength of the traveling wave, precisely in the form needed to eliminate ϵ from the ODE that F satisfies.*

Second: you should be able to integrate the third order ODE just obtained once, and reduce it to a second order ODE — with some arbitrary integration constant. If you multiply this second order ODE by $\dfrac{dF}{d\zeta}$, the result can again be integrated, so that you will end up with a first order ODE for F — with two integration constants in it.

> What does "to integrate an ODE" mean?
>
> *To integrate an ODE, you must first write it in the form* derivative (something) = 0*, from which you can conclude that* something = constant *— you have just "integrated" the ODE once. Sometimes an ODE cannot be written directly as the derivative of something, but it is still possible to do so by multiplying the equation by an appropriate* integrating factor*. This is what the multiplication by $dF/d\zeta$ in this hint will accomplish.*

Finally, consider the limits as $\zeta \to \pm\infty$ of the ODE you obtained in the previous steps. Using the assumed properties for F, you should now be able to obtain that $u_0 = u_1$. Showing that you also need $V < u_0 = u_1$ for a solution to exists is a bit trickier, and will require you to do some analysis of the equation — similar to the one used to study the traveling wave solutions of (4.1).

4.1.3 First order ODE review

Here we review some facts about real valued solutions to ODE's of the form

$$\left(\frac{dy}{dx}\right)^2 = R(y)\,, \tag{4.9}$$

where R is a real valued, smooth function. We consider the case $\boldsymbol{R = (y-a)(y-b)^2}$ only, where a and b are constants. However, the same type of analysis can be used for R of general form, by considering the zeros of R — i.e.: the points y_* at which $R(y_*) = 0$. In particular, note that the case $R = c(y-a)(y-b)^2$, where $c > 0$ is a constant, can be reduced to this one by the simple scaling $x \to \sqrt{c}\,x$. We will also **assume that $a \neq b$**, since $a = b \implies y = a + \dfrac{4\delta}{(x-x_0)^2}$ (where x_0 is a constant and $\delta = 0$ or $\delta = 1$) and there is not much to analyze.

> The **process below is based in the following idea, useful for any R:** once we know the behavior of the solutions of (4.9) near the zeros of R, and for $|y|$ large, the overall qualitative behavior is easy to ascertain. This is because the sign of $\dfrac{dy}{dx}$ can change only at the zeros of R, so that $y = y(x)$ must be monotone between zeros.
>
> Imagine the solutions plotted in the $(x,\, y)$–plane — where we draw the horizontal lines $y = y_*$, for every zero of R. Now consider a solution in any horizontal strip between two such zeros: it will have to be monotone (increasing or decreasing) till it reaches one of the zeros. The key question is: can a zero be reached at a finite value of x? If not, then the solution will reach the zero **only** as either $x \to \infty$, or $x \to -\infty$. If yes, then we need to know what happens when the zero is reached, to continue the solution beyond it.
>
> It turns out that only first order zeros can be reached at a finite value of x, and there the solution "bounces" back from the zero, turning from monotone increasing to monotone decreasing (or vice versa), with a sign change in $\dfrac{dy}{dx}$. Thus first order zeros of R are associated with local maximums (or minimums) of the solutions.
>
> We must also study the solutions in the half planes between the largest (smallest) zero of R and ∞ (resp. $-\infty$). For this we need to know the solutions' behavior for $|y|$ large.

FIRST: It is clear that we **need** $y \geq a$. Else the right hand side in (4.9) is negative and the solution cannot be real valued — **except, possibly, for the trivial solution $y \equiv b$ (if $b < a$).**

SECOND: Consider the behavior of the non-trivial solutions near the zeros of R (i.e.: exclude the solutions $y \equiv a$ and $y \equiv b$).

- **For $y \approx a$**, write $y = a + z(x)$, with $z > 0$ small. Then $(dz/dx)^2 = (a-b)^2\, z$ approximates the equation. This has the solutions $z = \dfrac{1}{4}(a-b)^2(x-x_0)^2$, where x_0 is a constant. Thus it is clear that the solutions of (4.9) will reach local minimums when y approaches a.
- **For $y \approx b$**, write $y = b + z(x)$, with z small. Then $(dz/dx)^2 = (b-a)\, z^2$ approximates the equation. This has real solutions only if $b > a$, with $z = c\, e^{\kappa x}$, c a constant and $\kappa = \pm\sqrt{b-a}$. Thus the solutions of (4.9) can approach b in the limits $x \to \pm\infty$, but only if $b > a$.

THIRD: Use these results to analyze the real valued solutions of (4.9) in the two possible cases.

Case $a > b$. The solutions are real only if $y \geq a$ (except for the trivial solution $y \equiv b$). Then either $y \equiv a$ or $y > a$ somewhere. In this second case the solution has a minimum at some $x = x_0$ with $(dy/dx) > 0$ for $x > x_0$ and $(dy/dx) < 0$ for $x < x_0$. Away from x_0, the solution grows without bound. Eventually y becomes very large, $(dy/dx)^2 \approx y^3$, and the solutions blow up like $4/(x-x_*)^2$. **Thus the only bounded real solutions are the trivial ones $y \equiv a$ and $y \equiv b$.**

Case $b > a$. If $y > b$ anywhere, the solution decays to $y = b$ as either $x \to \pm\infty$, with a singularity like $4/(x - x_*)^2$ at a finite $x = x_*$. This follows because either $(dy/dx) > 0$ or $(dy/dx) < 0$ and there is no way for the sign to change. On the other hand, if $a < y < b$, the solution decreases from $y = b$ at $x = -\infty$, to a minimum at some $x = x_0$ (where $y = a$) and then increases back to $y = b$ at $x = \infty$. **In this case nontrivial bounded solutions exist in the range $a \le y < b$.** These have limits $y = b$ at $x = \pm\infty$ and a single minimum (where $y = a$) at some finite $x = x_0$.

The analysis above depends only on the nature of the zeros of R. For $R = (y - a)(y - b)^2$ we can **solve the equation explicitly** and verify the results: Introduce $z = z(x)$ by $z^2 = y - a$ and write $\nu^2 = b - a$. Equation (4.9) — i.e.: $(y')^2 = (y - a)(y - b)^2$ — then becomes

$$4z^2(z')^2 = z^2(z^2 - \nu^2)^2 \iff 2z' = \pm(z^2 - \nu^2) \iff \ln\left(\frac{z - \nu}{z + \nu}\right) = \pm\nu x + \alpha,$$

where the prime denotes differentiation with respect to x and α is an arbitrary constant. Here we do not make any assumptions on the signs of $y - a$ and $b - a$: **thus z, ν and α need not be real.** With some further manipulation this yields, in terms of λ (defined by $\lambda^2 = -\exp(\pm\nu x + \alpha)$),

$$z = \nu\frac{1 - \lambda^2}{1 + \lambda^2} \iff \nu^2 - z^2 = \nu^2\left(\frac{2}{\lambda + \lambda^{-1}}\right)^2 \iff y = b - \nu^2\left(\frac{2}{\lambda + \lambda^{-1}}\right)^2.$$

Finally we can write, being now careful with keeping things real:

Case $a > b$. Let $\mu > 0$ be defined by $\mu^2 = a - b$. Then the nontrivial real solutions are

$$y = b + \mu^2 \sec^2\left(\frac{1}{2}\mu(x - x_0)\right), \tag{4.10}$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$). The relationship with the constants defined earlier is $\nu = \pm i\mu$ and $\alpha \pm \nu x_0 = i\pi$. See the left frame in figure 4.1.

Case $b > a$. Let $\mu > 0$ be defined by $\mu^2 = b - a$. Then the nontrivial real solutions are

$$y = b - \mu^2 \operatorname{sech}^2\left(\frac{1}{2}\mu(x - x_0)\right) \quad \text{and} \quad y = b + \mu^2 \operatorname{cosech}^2\left(\frac{1}{2}\mu(x - x_0)\right), \tag{4.11}$$

where x_0 is a constant (the trivial solutions are $y \equiv a$ and $y \equiv b$.) The relationship with the constants defined earlier is $\nu = \pm\mu$ and either $\alpha \pm \nu x_0 = i\pi$ or $\alpha \pm \nu x_0 = 0$. See the right frame in figure 4.1.

Next we briefly summarize the **situation for more general forms of** R — which can be analyzed with the same techniques we used here. For non-trivial real valued solutions:

1. At a first order zero of R, the solutions will either achieve a local minimum — if $dR/dy > 0$ there, or a local maximum — if $dR/dy < 0$ there. In other words, solutions can achieve values that correspond to simple zeros of R at some finite value of x, where they will have a local maximum or a local minimum. But these values cannot occur in the limits $x \to \pm\infty$.
2. By contrast, solutions cannot achieve values that correspond to higher (bigger than one) order zeros of R, at any finite value of x. Such values can be achieved only in the limits where $x \to \pm\infty$. For double zeros of R, the approach to the value (as $x \to \pm\infty$) is exponential. For zeros of order higher than two, the approach is algebraic .
3. Periodic oscillatory solutions occur in the regions between simple zeros, where $R > 0$. This situation does not occur in the example treated earlier, which has a single simple zero.
4. Single bump (or dip) solutions occur in the regions, comprised between a simple and a double (or higher) order zero, where $R > 0$. The right frame in figure 4.1 shows an example of this.

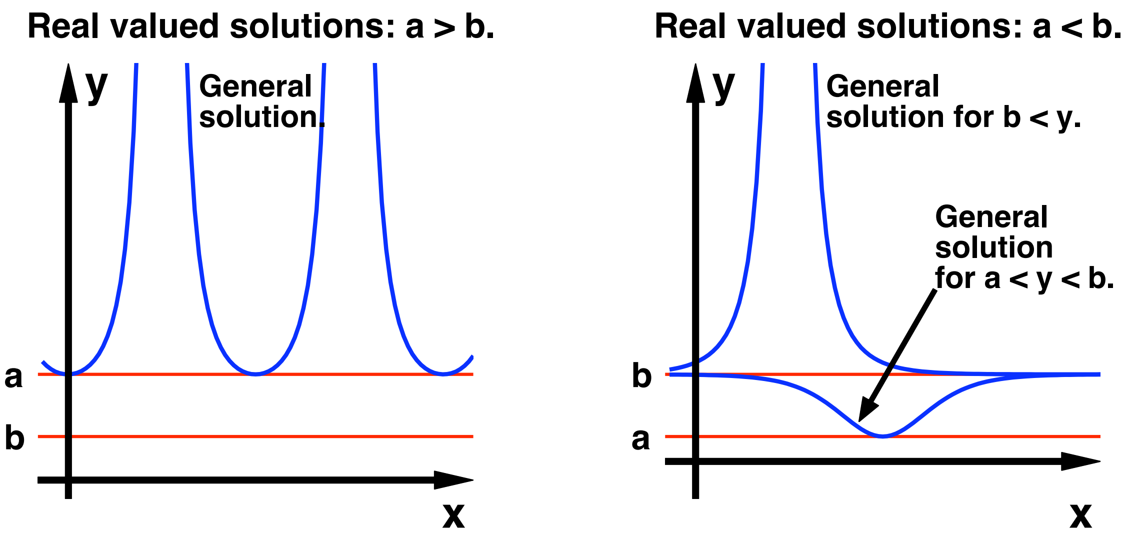

Figure 4.1: Real valued solutions of equation (4.9), with $R = (y - a)(y - b)^2$.
- Left frame: case $a > b$. The non-trivial solutions are periodic, with period $T = 2\pi/\sqrt{a - b}$.
- Right frame: case $a < b$. All the solutions satisfy $y \to b$ as $|x| \to \infty$.

5. The behavior of the solutions in regions where $R > 0$, comprised between a zero of R and $\pm\infty$, depends on the behavior of $R(y)$ for large values of y. If this leads to the formation of singularities (e.g.: $R = R(y)$ grows faster than linear as $y \to \pm\infty$), then the solutions are as in figure 4.1: either a periodic array of singularities (for a simple zero, left frame) or a single singularity — with decay to the value of y at the zero of R — as $x \to \pm\infty$ (for a double or higher order zero, right frame.)

4.2 Answer: Solitary wave for the KdV equation

The objective here is to find an expression for the **traveling wave** solutions of (4.6). These are solutions which are independent of time in a moving coordinate frame — i.e.: solutions of the form (4.7). Substituting (4.7) into (4.6) we obtain an ODE for the function $F = F(\zeta)$. Namely:

$$\frac{d^3F}{d\zeta^3} = F\frac{dF}{d\zeta} - V\frac{dF}{d\zeta}, \tag{4.12}$$

where the scaling by ϵ of the argument of F has the effect of eliminating the parameter ϵ from the equation for F. This means that the traveling waves for (4.6) all have the same functional form (shape), independent of the value of ϵ. The only role ϵ plays is in determining their wavelength.

Equation (4.12) can be integrated twice. The first integration is obvious and yields:

$$\frac{d^2F}{d\zeta^2} = \kappa + \frac{1}{2}F^2 - VF, \quad \text{where } \kappa \text{ is a constant.} \tag{4.13}$$

Next we multiply this by $2\,\dfrac{dF}{d\zeta}$ and integrate again, to obtain

$$\left(\frac{dF}{d\zeta}\right)^2 = \delta + 2\,\kappa\,F + \frac{1}{3}F^3 - V\,F^2\,, \quad \text{where } \delta \text{ is a constant.} \tag{4.14}$$

The general solution to (4.14) can be written in terms of elliptic functions. However, we are only interested in a very special type of traveling wave, connecting two states u_0 and u_1 at $\zeta = \pm\infty$. That is, all the derivatives of F vanish as $\zeta \to \pm\infty$, and F has limits $F \to u_0$ as $\zeta \to -\infty$ and $F \to u_1$ as $\zeta \to \infty$. Implementing these conditions in (4.13) and (4.14) we obtain

$$\left.\begin{array}{rcccl} 0 &=& \kappa - V\,u_0 + \frac{1}{2}\,u_0^2 &=& \kappa - V\,u_1 + \frac{1}{2}\,u_1^2, \\ 0 &=& \delta + 2\kappa\,u_0 - V\,u_0^2 + \frac{1}{3}\,u_0^3 &=& \delta + 2\kappa\,u_1 - V\,u_1^2 + \frac{1}{3}\,u_1^3. \end{array}\right\} \tag{4.15}$$

From these equations we can write (after a bit of algebra)

$$u_0 = u_1 = u_\infty, \quad \kappa = V\,u_\infty - \frac{1}{2}\,u_\infty^2 \quad \text{and} \quad \delta = -V\,u_\infty^2 + \frac{2}{3}\,u_\infty^3, \tag{4.16}$$

where $\boldsymbol{u_\infty}$ is the **common** value for $\boldsymbol{u_0}$ and $\boldsymbol{u_1}$. Right away we see that **no "shock-like"** (connecting two distinct states at $\pm\infty$) **traveling waves are possible.**

Substituting (4.16) into (4.14) the equation for the traveling wave shape F takes the form

$$\left(\frac{dF}{d\zeta}\right)^2 = \frac{1}{3}\,(F - u_\infty)^2\,(F - 3\,V + 2\,u_\infty)\,. \tag{4.17}$$

This equation is easy to analyze qualitatively (see § 4.1.3). Clearly, F can only take values satisfying $3V - 2u_\infty \leq F$ (otherwise the right hand side fails to be non–negative). Furthermore, if $3V - 2u_\infty > u_\infty$, then it is easy to see that all the solutions are singular (they "blow up" at a finite ζ). In fact, to have a non-trivial real valued solution F (with the appropriate limits as $\zeta \to \pm\infty$) we need $\boldsymbol{3V - 2u_\infty < F < u_\infty}$. This (in particular) implies $\boldsymbol{V < u_\infty}$. Other than this, there are no additional restrictions. It follows that

> **The shape of the traveling wave is completely determined by the two constants $\boldsymbol{V}$ (the speed) and $\boldsymbol{u_\infty}$ (the common value of the solution at $\boldsymbol{\zeta = \pm\infty}$), which are arbitrary except for $\boldsymbol{V < u_\infty}$.** (4.18)

It is possible to write the solution of (4.17) in terms of elementary functions. Introduce $\mu > 0$ by $\boldsymbol{\mu^2 = 3(u_\infty - V)}$ and let $\boldsymbol{G = u_\infty - F}$. Then $0 < G < \mu^2$, and G satisfies the equation

$$\left(\frac{dG}{d\zeta}\right)^2 = -\frac{1}{3}\,G^2\,(G - \mu^2), \tag{4.19}$$

where G vanishes at $\zeta = \pm\infty$. We solve this by separation of variables, as follows:

$$\frac{\sqrt{3}\,dG}{G\sqrt{\mu^2 - G}} = \pm d\zeta \iff \frac{-2\sqrt{3}\,dH}{\mu^2 - H^2} = d\zeta \iff \ln(\mu - H) - \ln(\mu + H) = \frac{1}{\sqrt{3}}\,\mu\,(\zeta - \zeta_0)\,,$$

where $\boldsymbol{H = \pm\sqrt{\mu^2 - G}}$ (thus $-\mu < H < \mu$) and ζ_0 is the constant of integration.[6] Exponentiating this last expression we get $H = \mu(1 - e^{\frac{1}{\sqrt{3}}\mu(\zeta - \zeta_0)})/(1 + e^{\frac{1}{\sqrt{3}}\mu(\zeta - \zeta_0)})$, which yields

$$G = \mu^2 - H^2 = 4\mu^2 e^{\frac{1}{\sqrt{3}}\mu(\zeta - \zeta_0)}\left(1 + e^{\frac{1}{\sqrt{3}}\mu(\zeta - \zeta_0)}\right)^{-2} = \mu^2 \mathrm{sech}^2\left(\frac{\mu}{2\sqrt{3}}(\zeta - \zeta_0)\right). \tag{4.20}$$

Thus we obtain, in terms of the two arbitrary real parameters u_∞ and $\mu > 0$, the solution

$$u = u_\infty - \mu^2 \mathrm{sech}^2\left(\frac{\mu}{2\sqrt{3}}\,\boldsymbol{\Theta}\right), \quad \text{where } V = u_\infty - \frac{1}{3}\mu^2 \text{ and } \boldsymbol{\Theta} = \frac{1}{\epsilon}\,(x - x_0 - V\,t), \tag{4.21}$$

where we have absorbed the constant ζ_0 into the constant x_0, and u solves the KdV equation (4.6).

[6] ζ_0 is just a shift in the position of the wave, allowed because the original equations are invariant under translation.

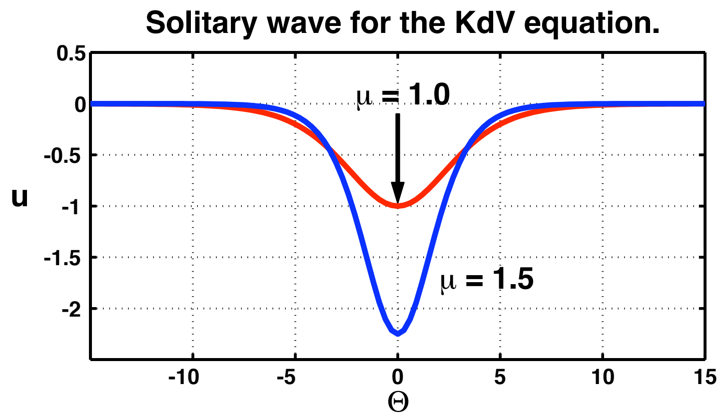

Figure 4.2: KdV equation (4.6) solitary wave solutions, given by (4.21), for $\mu = 1$ & $\mu = 1.5$, both with $u_\infty = 0$.

The solution in (4.21) is a traveling "dip", with the minimum moving along $x = x_0 + V\,t$. This kind of solution is called a **solitary wave**. Figure 4.2 shows two examples of this solution for $\mu = 1$ and $\mu = 1.5$, both with $u_\infty = 0$. Note that the parameter μ measures the degree of nonlinearity in the solitary wave. The larger μ is, the sharper and deeper the dip in the solution is.

Supplementary Materiel

5 1-D isentropic Euler equations of Gas Dynamics (supplementary)

Equations that govern the behavior of a gas can be derived using the same *Conservation Equation* techniques used to derive equations for the examples of Traffic Flow, River Flows, Shallow Water Waves, Modulations of Dispersive Waves, etc. The conserved quantities in this case are the mass, the momentum and the energy. The resulting equations are the **Euler equations of Gas Dynamics.**

Under certain conditions, one can assume that the entropy is a constant throughout the flow. Then the equations can be simplified, with the elimination of the equation for the conservation of energy. The **one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics** are

$$\left.\begin{array}{rcl} \rho_t + (\rho\,u)_x & = & 0, \\ (\rho\,u)_t + (\rho\,u^2 + p)_x & = & 0, \end{array}\right\} \tag{5.22}$$

where $\rho = \rho(x,\,t)$, $u = u(x,\,t)$, and $p = p(x,\,t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation here implements the conservation of mass, and the second the conservation of momentum. We also need an equation relating the fluxes of the conserved quantities with the conserved densities — the analogue of the equation $q = Q(\rho)$ in Traffic Flow. In this case this is provided by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \tag{5.23}$$

For example, for an ideal gas $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

The system of equations given by (5.22) is known as the

Conservation Form of the Isentropic Equations of Gas Dynamics, in Eulerian Coordinates.

Remark 5.4 *Normally, the pressure is a function of both the density and some other thermodynamic variable, such as the temperature. But the isentropic assumption allows us to write the pressure as a function of the gas mass density only.* ♣

Introduce now the function $\boldsymbol{c = c(x,\, t) > 0}$ (*c* **is the sound speed**) by[7]

$$c = C(\rho), \quad \text{where } C(\rho) = \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}(\rho)}. \tag{5.24}$$

An alternative form of the equations in (5.22) is then given by

$$\left.\begin{array}{rcl} \rho_t + u\,\rho_x + \rho\,u_x &=& 0, \\ u_t + \dfrac{c^2}{\rho}\,\rho_x + u\,u_x &=& 0. \end{array}\right\} \tag{5.25}$$

Multiplying the first equation here by c/ρ, and adding (or subtracting) it to the second, yields the **characteristic form** of the equations

$$0 = (u_t + (u+c)u_x) + \frac{c}{\rho}\,(\rho_t + (u+c)\rho_x)\,, \tag{5.26}$$

$$0 = (u_t + (u-c)u_x) - \frac{c}{\rho}\,(\rho_t + (u-c)\rho_x)\,. \tag{5.27}$$

Equivalently

$$0 = \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{c}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + c, \tag{5.28}$$

$$0 = \frac{\mathrm{d}u}{\mathrm{d}t} - \frac{c}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u - c. \tag{5.29}$$

These last two equations show that, if we **introduce a new variable $\boldsymbol{h = h(\rho)}$**, defined by

$$\frac{\mathrm{d}h}{\mathrm{d}\rho} = \frac{C(\rho)}{\rho}, \tag{5.30}$$

then

$$\boldsymbol{u + h} \quad \textbf{is constant along the characteristics} \quad \boldsymbol{\frac{\mathrm{d}x}{\mathrm{d}t} = u + c,} \tag{5.31}$$

$$\boldsymbol{u - h} \quad \textbf{is constant along the characteristics} \quad \boldsymbol{\frac{\mathrm{d}x}{\mathrm{d}t} = u - c.} \tag{5.32}$$

The variable $\boldsymbol{\mathcal{R} = u + h}$ is the **right Riemann invariant**, while $\boldsymbol{\mathcal{L} = u - h}$ is the **left Riemann invariant.**

THE END.

[7] For an ideal gas, $\boldsymbol{c = \sqrt{\gamma\, p/\rho}}$. For dry air at one atmosphere and 15 degrees Celsius: $p = 1.013 \times 10^6$ dyn/cm^2, $\rho = 1.226 \times 10^{-3}$ g/cm^3, and $\gamma = 1.401$. Hence $c = 340.2$ m/s — measured value is $c = 340.6$ m/s.