# Answers to P-Set # 04, 18.300 MIT (Spring 2022)

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# 1 Haberman 7401. Solve initial value problem

## 1.1 Statement: Solve initial value problem

Assume that $u(\rho) = u_m\,(1 - \rho/\rho_j)$, where $u_m$ is the speed limit and $\rho_j$ is the jamming density. For the initial conditions:

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \ x < 0, \\ \rho_0\,(L - x)/L & \text{for } \ 0 \le x \le L, \\ 0 & \text{for } \ L < x, \end{cases} \tag{1.1}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, determine and sketch $\rho(x,\,t)$.

### 1.2 Answer: Solve initial value problem

Note that $c = c(\rho) = \dfrac{\mathrm{d}(\rho\, u)}{\mathrm{d}\rho} = u_m \left(1 - \dfrac{2\,\rho}{\rho_j}\right)$ is a **decreasing** function of $\rho$. We now solve using characteristics, as follows:

**Region (1)** **($x < 0$ at $t = 0$).** Here $\rho = \rho_0$ along $x = c_0\, t + \zeta$, where $\zeta < 0$ and $c_0 = c(\rho_0)$, with $c_0 < u_m$. Eliminating $\zeta$, it follows that ...................... $\boldsymbol{\rho = \rho_0}$ **for** $\boldsymbol{x < c_0\, t}$.

**Region (2)** **($0 \le x \le L$ at $t = 0$).** Here $\rho = \dfrac{\rho_0\,(L - \zeta)}{L}$ along $x = c\left(\dfrac{\rho_0\,(L-\zeta)}{L}\right) t + \zeta$, where $0 \le \zeta \le L$.

Eliminating $\zeta$, it follows that ...................... $\boldsymbol{\rho = \dfrac{u_m\, t + L - x}{(u_m - c_0)\, t + L}\, \rho_0}$ **for** $\boldsymbol{c_0\, t \le x \le u_m\, t + L}$.

**Region (3)** **($L < x$ at $t = 0$).** Here $\rho = 0$ along $x = u_m\, t + \zeta$, where $L < \zeta$. Eliminating $\zeta$, it follows that ................................. $\boldsymbol{\rho = 0}$ **for** $\boldsymbol{u_m\, t + L < x}$.

Summarizing, we have (see figure 1.1)

$$\rho(x,\, t) = \begin{cases} \rho_0 & \text{for } x < c_0\, t. \\ \dfrac{u_m\, t + L - x}{(u_m - c_0)\, t + L}\, \rho_0 & \text{for } c_0\, t \le x \le u_m\, t + L. \\ 0 & \text{for } u_m\, t + L < x. \end{cases} \tag{1.2}$$

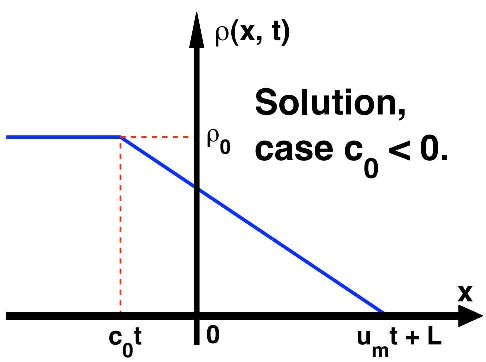


Figure 1.1: Haberman 7401. Solution to the initial value problem posed in equation (1.1), for the case $c(\rho_0) < 0$, plotted for some arbitrary $t > 0$. The case $c(\rho_0) > 0$ is similar.

# 2 Haberman 7402. Solve initial value problem

### 2.1 Statement: Solve initial value problem

Assume that $u(\rho) = u_m\,(1 - \rho^2/\rho_j^2)$, where $u_m$ is the speed limit and $\rho_j$ is the jamming density. For the initial conditions:

$$\rho(x,\, 0) = \begin{cases} \rho_0 & \text{for } x < 0, \\ \rho_0\,(L - x)/L & \text{for } 0 < x < L, \\ 0 & \text{for } L < x, \end{cases} \tag{2.1}$$

where $0 < \rho_0 < \rho_j$ and $0 < L$, **determine and sketch** $\rho(x,\, t)$**.**

## 2.2 Answer: Solve initial value problem

Note: $c = c(\rho) = \dfrac{\mathrm{d}(\rho\, u)}{\mathrm{d}\rho} = u_m \left(1 - \dfrac{3\,\rho^2}{\rho_j^2}\right)$ is a **decreasing** function of $\rho$. We solve using characteristics:

**Region (1) ($x < 0$ at $t = 0$).** Here $\rho = \rho_0$ along $x = c_0\, t + \zeta$,
where $\zeta < 0$ and $c_0 = c(\rho_0)$, with $c_0 < u_m$. Eliminating $\zeta$, it follows that ......... $\boldsymbol{\rho = \rho_0}$ **for** $\boldsymbol{x < c_0\, t}$.

**Region (2) ($0 \le x \le L$ at $t = 0$).** Here $\rho = \dfrac{\rho_0\,(L-\zeta)}{L}$ along $x = c\left(\dfrac{\rho_0\,(L-\zeta)}{L}\right) t + \zeta$,
where $0 \le \zeta \le L$.

Eliminating $\zeta$, it follows that ........................ $\boldsymbol{\rho = \dfrac{-L\,\rho_0}{2\,t\,(u_m - c_0)}\left(1 - \sqrt{1 + \dfrac{4\,t\,(u_m - c_0)\,\lambda}{L^2}}\right)}$,

where $\boldsymbol{\lambda = u_m\, t + L - x}$, and $c_0\, t \le x \le u_m\, t + L$.

**Region (3) ($L < x$ at $t = 0$).** Here $\rho = 0$ along $x = u_m\, t + \zeta$,
where $L < \zeta$. Eliminating $\zeta$, it follows that ................................ $\boldsymbol{\rho = 0}$ **for** $\boldsymbol{u_m\, t + L < x}$.

Summarizing, we have (see figure 2.1)

$$\rho(x,\,t) = \begin{cases} \rho_0 & \text{for } x < c_0\, t, \\ \dfrac{-L\,\rho_0}{2\,t\,(u_m - c_0)}\left(1 - \sqrt{1 + \dfrac{4\,t\,(u_m - c_0)\,\lambda}{L^2}}\right) & \text{for } c_0\, t \le x \le u_m\, t + L, \\ 0 & \text{for } u_m\, t + L < x, \end{cases} \tag{2.2}$$

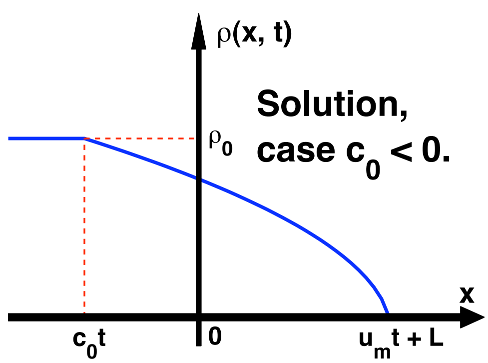


Figure 2.1: Haberman 7402. Solution the the initial value problem posed in equation (2.1), for $c(\rho_0) < 0$, plotted for some arbitrary $t > 0$. The case $c(\rho_0) > 0$ is similar.

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# 3 Haberman 7701. Shock velocity when $u = u(\rho)$ is linear

## 3.1 Statement: Shock velocity when $u = u(\rho)$ is linear

If $u = u_{\max}\,(1 - \rho/\rho_{\max})$, **what is the velocity of a traffic shock separating the densities $\rho_0$ and $\rho_1$?** Simplify the expression as much as possible. Show that the shock velocity is the average of the density wave velocities associated with $\rho_0$ and $\rho_1$.

### 3.2 Answer: Shock velocity when $u = u(\rho)$ is linear

The shock speed $s$ is given by the formula

$$s = \frac{[\,q\,]}{[\,\rho\,]} = \frac{q_1 - q_0}{\rho_1 - \rho_0}, \tag{3.1}$$

where $q = \rho\, u = \rho\, u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$, $q_0 = q(\rho_0)$, and $q_1 = q(\rho_1)$.

We note that $q = q(\rho)$ is a **quadratic** function of $\rho$. Thus, we can write:

$$q_1 = q_0 + c_0\,(\rho_1 - \rho_0) + \frac{1}{2}\,d_0\,(\rho_1 - \rho_0)^2, \tag{3.2}$$

$$c_1 = c_0 + d_0\,(\rho_1 - \rho_0), \tag{3.3}$$

where $\boldsymbol{c = c(\rho) = \frac{dq}{d\rho}}$, $\boldsymbol{c_0 = c(\rho_0)}$, $\boldsymbol{c_1 = c(\rho_1)}$, and $\boldsymbol{d_0 = \frac{d^2q}{d\rho^2}(\rho_0)}$. Substituting these expansions into the equation for the shock speed above in (3.1), we find:

$$s = c_0 + \frac{1}{2}\,d_0\,(\rho_1 - \rho_0) = \frac{1}{2}\,(c_0 + c_1). \tag{3.4}$$

Since $c = u_{\mathsf{max}}\,(1 - 2\,\rho/\rho_{\mathsf{max}})$, this last equation can also be written in the form:

$$s = u_{\mathsf{max}} - \frac{u_{\mathsf{max}}}{\rho_{\mathsf{max}}}\,(\rho_0 + \rho_1). \tag{3.5}$$

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# 4 Haberman 7902. Shock velocity

### 4.1 Statement: Shock velocity

Suppose that

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \tag{4.1}$$

Determine the velocity of the shock. Briefly give a physical explanation of the result. What does this shock correspond to?

### 4.2 Answer: Shock velocity

Since $c$ is a decreasing function of $\rho$, $c(0) > c(\rho_0)$. Thus these initial conditions give rise to a shock, with speed:

$$s = \frac{q(\rho_0) - q(0)}{\rho_0 - 0} = \frac{q(\rho_0)}{\rho_0} = u(\rho_0). \tag{4.2}$$

This should not be surprising. The shock is the position of the last car in a uniform stream of traffic at density $\rho_0$. Obviously, this car moves at speed $u(\rho_0)$.

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# 5 KiNe03. Initial value problem with $Q$ quadratic

### 5.1 Statement: Initial value problem with $Q$ quadratic

Consider the traffic flow equation

$$\rho_t + q_x = 0, \tag{5.1}$$

for a flow $q = Q(\rho)$ that is a quadratic function of $\rho$. In this case $c = \mathrm{d}Q/\mathrm{d}\rho$ is a conserved quantity as well (why?). Thus the problem (including shocks, if any) can be entirely formulated in terms of $c$, which satisfies

$$c_t + \left(\frac{1}{2}\,c^2\right)_x = 0. \tag{5.2}$$

**1.** Consider the initial value problem determined by (5.2) and [1]

$$c(x,\,0) = 0 \;\text{ for }\; x \le 0 \quad\text{and}\quad c(x,\,0) = 2\sqrt{x} \ge 0 \;\text{ for }\; x \ge 0. \tag{5.3}$$

Without actually solving the problem, **argue that the solution to this problem must have the form**

$$c = t\,f(x/t^2) \;\text{ for }\; t > 0, \quad \text{for some function } f. \tag{5.4}$$

*Hint.* Let $c = c(x,\,t)$ be the solution. For any constant $a > 0$, define $\mathcal{C} = \mathcal{C}(x,\,t)$ by $\boldsymbol{\mathcal{C} = \frac{1}{a}\,c(a^2\,x,\,a\,t)}$. What problem does $\mathcal{C}$ satisfy? Use now the fact that the solution to (5.2–5.3) is unique to show that (5.4) must apply, by selecting the constant $\boldsymbol{a}$ appropriately at any fixed time $t > 0$. ♣

**2.** Use the method of characteristics to **solve the problem in (5.2–5.3).** Write the solution **explicitly** for all $\boldsymbol{t > 0}$, and **verify that it satisfies (5.4).** *Warning: the solution involves a square root. Be careful to select the correct sign, and to* **justify your choice.**

**3.** For the solution obtained in item **2**, **evaluate $\boldsymbol{c_x}$ at $\boldsymbol{x = 0}$ for $\boldsymbol{t > 0}$.** Note that this derivative is discontinuous there, so it has two values (left and right).

## 5.2 Answer: Initial value problem with $Q$ quadratic

First (note: this was not part of the problem), **why is $\boldsymbol{c}$ conserved?** The reason is that $c$ has the form $c = \alpha + \beta\,\rho$, for some constants $\alpha$ and $\beta$. Hence $\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b c\,\mathrm{d}x = \beta\,\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho\,\mathrm{d}x = \beta\,q_a - \beta\,q_b$ for any interval $[a, b]$. ♣

Now we proceed with the answer to the problem. Note that

$$\boldsymbol{\mathcal{C}_t = c_t} \quad\text{and}\quad \boldsymbol{(\mathcal{C}^2)_x = (c^2)_x}, \tag{5.5}$$

where $c$ and its derivatives evaluated at $(a^2\,x,\,a\,t)$. Further: $\mathcal{C}(x,\,0) = c(x,\,0)$. It follows that $\mathcal{C} = c$, that is:

$$\boldsymbol{c(x,\,t) = \frac{1}{a}\,c(a^2\,x,\,a\,t)} \quad\text{for any}\quad \boldsymbol{a > 0}. \tag{5.6}$$

Now, evaluate (5.6) at $t = 1/a$.
Since $a > 0$ is arbitrary, it follows that

$$\boldsymbol{c(x,\,t) = t\,c(x/t^2,\,1)} \quad\text{for any}\quad \boldsymbol{t > 0}, \tag{5.7}$$

which is (5.4) with $\boldsymbol{f(\xi) = c(\xi,\,1)}$.

Next we solve (5.2–5.3) using characteristics. For $\zeta \le 0$ we obtain $c = 0$ along $x = \zeta$. Hence these characteristics give

$$\boldsymbol{c = 0} \;\;\textbf{for}\;\; \boldsymbol{x \le 0}. \tag{5.8}$$

On the other hand, for $\boldsymbol{\zeta \ge 0}$ the characteristics give [2] $\boldsymbol{c = 2\sqrt{\zeta}}$ along $\boldsymbol{x = 2\sqrt{\zeta}\,t + \zeta}$. Thus †

$$\boldsymbol{c = 2\left(\sqrt{x + t^2} - t\right) = 2\,t\left(\sqrt{1 + \frac{x}{t^2}} - 1\right)} \;\;\textbf{for}\;\; \boldsymbol{x \ge 0}. \tag{5.9}$$

† As $\boldsymbol{\zeta}$ varies from $\boldsymbol{\zeta = 0}$ to $\boldsymbol{\zeta = \infty}$, the characteristics $\boldsymbol{x = 2\sqrt{\zeta}\,t + \zeta}$ cover the entire region $\boldsymbol{x \ge 0}$. Further, they do so one-to-one, since $\boldsymbol{\partial_\zeta x = 1 + t/\sqrt{\zeta} > 0}$. Hence we can solve for $\boldsymbol{\zeta}$ as a function of $\boldsymbol{(x,\,t)}$. To do so we write these characteristics in the form $\boldsymbol{(t + \sqrt{\zeta})^2 = x + t^2}$, so that $\boldsymbol{\sqrt{\zeta} = -t + \sqrt{x + t^2}}$. Note that, since $\boldsymbol{\sqrt{\zeta} \ge 0}$ is required, the positive square root $\boldsymbol{\sqrt{x + t^2}}$ must be selected.

**The solution to (5.2–5.3) is given by (5.8–5.9). This clearly satisfies (5.4),** with

$$\boldsymbol{f(z) = 0} \;\;\textbf{for}\;\; \boldsymbol{z < 0}, \quad\textbf{and}\quad \boldsymbol{f(z) = 2\left(\sqrt{1 + z} - 1\right)} \;\;\textbf{for}\;\; \boldsymbol{z > 0}. \tag{5.10}$$

Finally, from (5.8–5.9), at $x = 0$ and $t > 0$,

$$\boldsymbol{c_x = 0} \text{ from the left, and } \boldsymbol{c_x = 1/t} \text{ from the right.} \tag{5.11}$$

[1] In a traffic problem, $\boldsymbol{c}$ must satisfy $\boldsymbol{c(\rho_j) \le c \le c(0)}$. Ignore the fact that this does not apply for (5.2).
[2] Note that here we take $\boldsymbol{\sqrt{\zeta} > 0}$ to match the initial data for $\boldsymbol{c}$.

Note that, as $t \to 0$, $c_x \to \infty$ on the right (which matches the initial data).

# 6 Linear 1st order PDE (problem 09)

## 6.1 Statement: Linear 1st order PDE (problem 09)

**Surface Evolution.** The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its "height" $h = h(x,\, y,\, t)$ relative to the $(x,\, y)$-plane of reference. Under appropriate conditions, a rather complicated pde can be written[3] for $h$. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r}\, h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \tag{6.1}$$

Axial symmetry is assumed, so that $h = h(r,\, t)$. Obviously, ***h* should be an even function of *r***. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r,\, 0) = H(r^2), \tag{6.2}$$

where $H$ is a smooth function describing a localized bump. Specifically: **(i)** $H(0) > 0$, **(ii)** $H$ is monotone decreasing. **(iii)** $H \to 0$ as $r \to \infty$. **Note that $h(r,\, 0)$** *is an even function of $r$.*

1. Using the theory of characteristics, write an explicit formula for the solution of (6.1 – 6.2).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2\,A$?
4. Show that the resulting solution is an even function of $r$ for all times.
5. Show that, as $t \to \infty$, the bump shrinks and vanishes. **Hint.** Pick some example function $H$ with the properties above, and plot the solution for various times. This will help you figure out why the bump shrinks and vanishes.

## 6.2 Answer: Linear 1st order PDE (problem 09)

The characteristic form of equation (6.1) is

$$\frac{dh}{dt} = 0 \quad \text{along the curves} \quad \frac{dr}{dt} = -\frac{A}{r}. \tag{6.3}$$

This yields

$$r = \sqrt{\zeta^2 - 2\,A\,t} \quad \text{and} \quad h = H(\zeta^2), \tag{6.4}$$

for the characteristic that starts (time $t = 0$) at $0 < r = \zeta < \infty$. The characteristics are parabolas pointing downward in space-time, with their "tips" along the time axis. When a characteristic reaches the origin, it exits the domain where the equation is valid, and it ends. See figure 6.1.

From the left equation in (6.4), we see that $\zeta^2 = r^2 + 2\,A\,t$. Thus the solution to the problem in (6.1 – 6.2) is

$$h = H(r^2 + 2\,A\,t). \tag{6.5}$$

[3] From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.

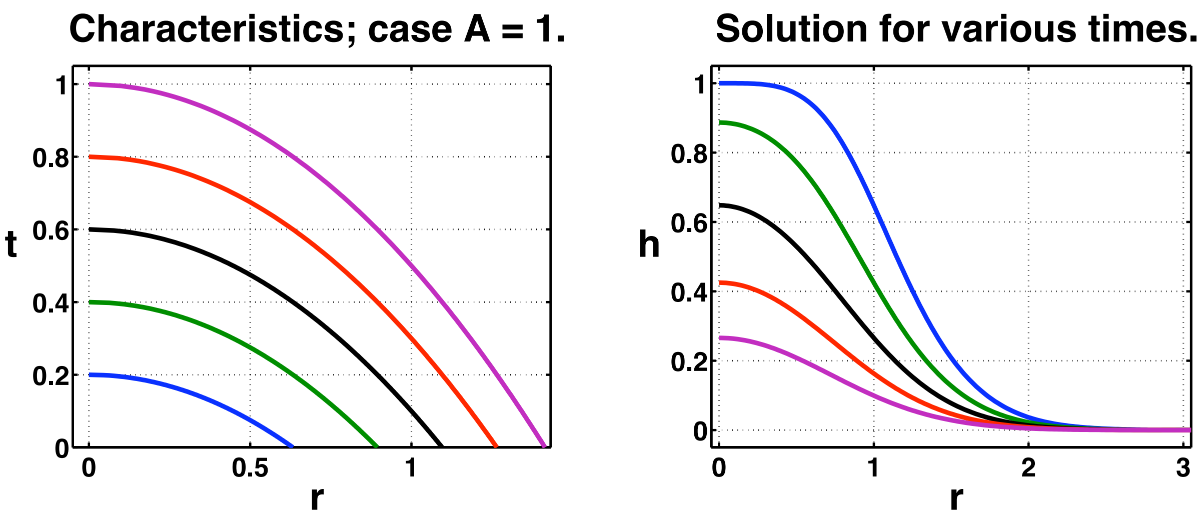


Figure 6.1: Linear 1st order pde #09 problem. Left: plot of a few typical characteristic curves for equation (6.1). Right: plots of the solution for $H(z) = \text{sech}(z)$, $A = 1$, and times (top to bottom) $t = 0,\ 1/4,\ 1/2,\ 3/4,\ 1.$

Clearly, *this is an even function of* $r$ *for all times.* Furthermore, since $H$ vanishes as it's argument goes to infinity, *the bump described by (6.5) shrinks and vanishes as* $t \to \infty$. See figure 6.1.

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**THE END.**