Answers to P-Set # 03, 18.300 MIT (Spring 2022)

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1 Haberman 6307. Design flow density curve fitting Lincoln Tunnel data

1.1 Statement: Design flow density curve fitting Lincoln Tunnel data

Consider exercise 6103.† Suppose that the drivers accelerate in such a fashion that

$$\alpha = u_t + u\,u_x = -\frac{a^2}{\rho}\rho_x, \quad \text{where } a > 0 \text{ is a constant.} \tag{1.1}$$

(a) Physically interpret this situation.

(b) If u only depends on ρ, and the equation for conservation of cars is valid, show that

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = -\frac{a}{\rho}. \tag{1.2}$$

(c) Solve the differential equation in part **(b)**, subject to the condition that $u(\rho_{\text{max}}) = 0$. The resulting flow–density curve fits quite well to the Lincoln Tunnel data.

(d) Show that a is the velocity that corresponds to the road's capacity.

(e) Discuss objections to the theory for small densities.

† Note: exercise 6103 asks to show that the car's acceleration is given by $\boldsymbol{\alpha = u_t + u\,u_x}$. Here just use this. Another useful result is that from exercise 6306, where the task is to show that .... $\boldsymbol{\alpha = -\rho\,\frac{du}{d\rho}\,u_x}$. This follows by: Substitute $\boldsymbol{u = u(\rho)}$ into $\boldsymbol{\alpha = u_t + u\,u_x}$. Then (chain rule) $\boldsymbol{\alpha = \frac{du}{d\rho}\,(\rho_t + u\,\rho_x) = \frac{du}{d\rho}\,(u - c)\,\rho_x}$ (use the equation in the last step). But $\boldsymbol{q = u\,\rho}$, so that $\boldsymbol{c = u + \frac{du}{d\rho}\,\rho}$. Hence $\boldsymbol{\alpha = -\rho\,(\frac{du}{d\rho})^2\,\rho_x}$. ♣

1.2 Answer: Design flow density curve fitting Lincoln Tunnel data

Notice that, from exercise 6103, we know that $\boldsymbol{\alpha}$ **above in equation (1.1) is the car acceleration.** Thus

(a) We can motivate (1.1) as follows:

• Drivers will accelerate if they see the density going down ahead of them and (conversely) slow down if the opposite is true. The simplest way to model this behavior is to make the car acceleration α proportional to the gradient of the density ρ_x, with a negative proportionality factor. That is, take $\alpha = -\kappa\rho_x$, for some $\kappa > 0$.

• We also expect the response of the drivers to the gradient of the density to be less intense as the density goes up (for "large" densities the velocities, hence the accelerations, will be small.) A simple way to model this is to take κ above inversely proportional to the density. That is, take: $\kappa = -a^2/\rho$, which then leads to (1.1).

(b) If u only depends on ρ (i.e.: $u = u(\rho)$), then we can use the result in exercise 63.6, namely: $\alpha = -\rho\,\dfrac{\mathrm{d}u}{\mathrm{d}\rho}\,u_x$. Thus:

$$-\frac{a^2}{\rho}\,\rho_x = \alpha = -\rho\,\frac{\mathrm{d}u}{\mathrm{d}\rho}\,u_x = -\rho\,\left(\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)^2\,\rho_x, \quad \Longrightarrow \quad \left(\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)^2 = \frac{a^2}{\rho^2}.$$

But $\dfrac{\mathrm{d}u}{\mathrm{d}\rho} < 0$, thus equation (1.2) follows from the last equality here.

(c) Equation (1.2) is an ode for $u = u(\rho)$, with general solution

$$u = -a\,\ln(\rho) + \text{constant}. \tag{1.3}$$

Since $u(\rho_{\text{max}}) = 0$, we can rewrite this in the form:

$$u = -a\,\ln\left(\frac{\rho}{\rho_{\text{max}}}\right). \tag{1.4}$$

This yields the flow–density relationship

$$q = \rho\,u = -a\,\rho\,\ln\left(\frac{\rho}{\rho_{\text{max}}}\right). \tag{1.5}$$

(d) From equation (1.5) it follows that

$$\frac{\mathrm{d}q}{\mathrm{d}\rho} = -a\,\left\{\ln\left(\frac{\rho}{\rho_{\text{max}}}\right) + 1\right\}. \tag{1.6}$$

Thus q achieves its maximum (**road capacity =** q_M) at $\rho_M = \frac{1}{e}\rho_{\mathsf{max}}$, with $q_M = a\rho_M$. In particular

$$u_M = a, \tag{1.7}$$

which gives an interpretation of ***a* as the car velocity at the maximum flow rate** (road's capacity.)

(e) Note that this theory gives unbounded car velocities as the density vanishes, so it has to be taken with some care then. This problem arises from the second assumption in **(a)**: while generally true that the drivers response to ρ_x will be more intense as the density goes down, it will certainly not go to infinity as ρ vanishes!

On the other hand, since one should not be using a continuum approximation for small densities anyway, this problem is not too troublesome.

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2 Problem 7104a. PDE satisfied by the wave velocity

2.1 Statement: PDE satisfied by the wave velocity

Consider the pde $\rho_t + q_x = \rho_t + c(\rho)\,\rho_x = 0$, where $q = q(\rho)$ and $c = c(\rho) = \frac{\mathrm{d}q}{\mathrm{d}\rho}(\rho)$. Assume that $\rho = \rho(x,\,t)$ is a smooth solution of the equation, and let $c = c(x,\,t) = c(\rho)$. **Find the pde that *c* satisfies**. Note: *the pde is the same no matter what solution ρ is selected! In this sense it is unique, hence the use of "Find the pde", not "Find a pde".*

2.2 Answer: PDE satisfied by the wave velocity

Multiply $\rho_t + c\,\rho_x = 0$ by $\frac{\mathrm{d}c}{\mathrm{d}\rho}$, and use the chain rule: $c_t = \frac{\mathrm{d}c}{\mathrm{d}\rho}\rho_t$ and $c_x = \frac{\mathrm{d}c}{\mathrm{d}\rho}\rho_x$. Hence $\boldsymbol{c_t + c\,c_x = 0}$. (2.1)

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3 Linear 1st order PDE (problem 03)

3.1 Statement: Linear 1st order PDE (problem 03)

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{3.1}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to find the solution. Write, **explicitly**, $\boldsymbol{u = u(x,\,y)}$ as a function of x and y, using f.

Hint. Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$ (where $-\infty < \tau < \infty$). Finally: eliminate τ, to get u as a function of x and y.

Part 2. In which part of the $(x,\,y)$ plane is the solution determined?

Hint. Draw in the x-y plane the characteristics computed in part 1.

Part 3. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

3.2 Answer: Linear 1st order PDE (problem 03)

Using x to parametrize the characteristic equations for the problem in (3.1), we obtain

$$dy/dx = 2\,x \quad \text{and} \quad du/dx = y, \tag{3.2}$$

with $y = \tau$ and $u = f(\tau)$ for $x = 0$ and $-\infty < \tau < \infty$. These equations have the solution

$$y = \tau + x^2 \quad \text{and} \quad u = f(\tau) + \tau\,x + \frac{1}{3}\,x^3. \tag{3.3}$$

Part 1. The first expression in (3.3) yields $\tau = y - x^2$. Then, from the second expression

$$u = f(y - x^2) + x\,(y - x^2) + \frac{1}{3}\,x^3. \tag{3.4}$$

Part 2. The characteristic curves, as given by (3.3), are parabolas intersecting the y-axis at $y = \tau$ — all shifted versions of the parabola $y = x^2$. Hence they cover the whole plane, with exactly one characteristic through every point. Thus **the solution is uniquely defined over the whole plane.**

Part 3. From (3.4) it should be clear that: *if f has a continuous derivative, then the partial derivatives of u are also continuous.*

4 TFPb09. Quasi-linear equation solution

4.1 Statement: Quasi-linear equation solution

Find the solution to

$$x^2\,\psi_x - \psi^2\,\psi_y = 0, \tag{4.1}$$

with $\boldsymbol{\psi = x}$ on $\boldsymbol{y = x}$, for $\boldsymbol{x > 0}$. **Where is the solution defined, and why?**

4.2 Answer: Quasi-linear equation solution

The characteristic equations for the problem in (4.1) are

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = 0 \quad \text{along the curves} \quad \frac{\mathrm{d}x}{\mathrm{d}s} = x^2 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}s} = -\psi^2, \tag{4.2}$$

with initial ($s = 0$) conditions $\psi = \xi$, $x = \xi$, and $y = \xi$ for $0 < \xi < \infty$. The solution is

$$x = \frac{\xi}{1 - \xi\,s}, \quad y = \xi - \xi^2\,s, \quad \text{and} \quad \psi = \xi. \tag{4.3}$$

Note that

- **(a)** As $s \to -\infty$, $x \to 0$ and $y \to \infty$.
- **(b)** As $s \to 1/\xi$, $x \to \infty$ and $y \to 0$.
- **(c)** The characteristics are defined for $-\infty < s < 1/\xi$ only, where $\xi > 0$. On them, **both:** $x,\ y > 0$. It should be easy to see that the characteristics are the family of hyperbolas in the first quadrant defined by $x\,y = \xi^2$.
- **(d)** The characteristics satisfy $x\,y = \xi^2$, and $\xi > 0$. Hence $\psi = \xi = \sqrt{x\,y}$ (recall: $x,\ y > 0$, see item **c**).

Summary: the characteristics are the family of hyperbolas $\boldsymbol{x\,y}$ = constant $\boldsymbol{> 0}$ in the first quadrant $\boldsymbol{(x,\ y > 0)}$, which they cover entirely. The solution to the problem, defined on the first quadrant, is $\boldsymbol{\psi = \sqrt{x\,y}}$.

5 TFPb10. Solve linear scalar pde using characteristics

5.1 Statement: Solve linear scalar pde using characteristics

Find the solution to the equation

$$x\,\phi_y + \phi_x = \phi, \tag{5.1}$$

with $\phi = y$ for $x = 0$ and $y > 0$.
Describe and plot the region where the characteristics reach (thus where the solution is defined).

5.2 Answer: Solve linear scalar pde using characteristics

The characteristic equations in this case are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}x} = \phi, \tag{5.2}$$

where $\phi = s$ and $y = s$ for $x = 0$, with $s > 0$.
Solving for the characteristics we obtain:

$$y = s + \frac{1}{2}x^2 \quad \text{and} \quad \phi = s e^x \quad (\text{for } s > 0) \Longrightarrow \phi = \left(y - \frac{1}{2}x^2\right) e^x, \tag{5.3}$$

where we used $s = y - \frac{1}{2}x^2$ to eliminate s in the expression for ϕ. Substitution into (5.1) shows that this solves the problem. Since $s = y - \frac{1}{2}x^2$ must be positive:
The solution in (5.3) is defined in the region $y > \frac{1}{2}x^2$ only — see figure 5.1.

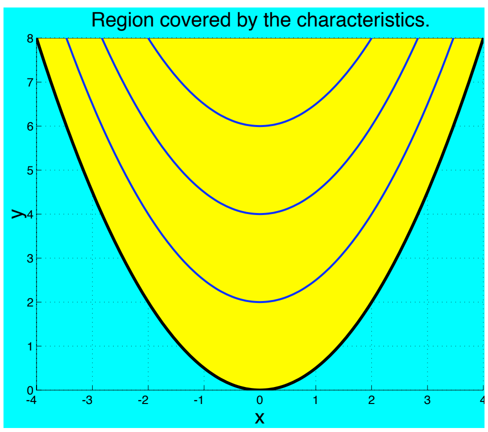

Figure 5.1: TFPb10. Region covered by the characteristics in the solution to equation (5.1) — yellow area. A few typical characteristics are shown.

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6 TFPb11. Check solution by implicit differentiation

6.1 Statement: Check solution by implicit differentiation

Consider the equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = -\rho. \tag{6.1}$$

A. Show by direct substitution that $\rho = \rho(x, t)$, as defined by the implicit equation

$$\rho = \exp(-t) F(x + \rho) \quad (\text{here } F = F(x) \text{ is an arbitrary function}), \tag{6.2}$$

solves equation (6.1) above. That is: first use implicit differentiation to calculate $\frac{\partial \rho}{\partial t}$ and $\frac{\partial \rho}{\partial x}$. Then substitute the answers to this calculation in the pde, and show directly that the equation is satisfied.
You may assume that ρ and F have as many derivatives as needed.

B. **Find the characteristics for equation (6.1) above. Then show how the solution in (6.2) follows from using them,** in the same fashion used in the lectures to show that the solution to the Traffic Flow equations can be written implicitly as $\rho = F(x - c(\rho)t)$. Recall that *the characteristics are special curves along which the pde reduces to an ode*

Remark 6.1 *Equation* (6.1) *cannot be a traffic flow equation, since it does not conserve the number of cars: cars are being removed from the road at a rate proportional to the local car density. Well, this could perhaps be explained by a road with lots of big potholes, but there is a bigger problem: the wave speed $c = \rho$ increases with the density ρ, while for traffic flow it should decrease.* ♣

6.2 Answer: Check solution by implicit differentiation

A. Taking derivatives of equation (6.2) we find:

$$\rho_t = -\exp(-t)F(\xi) + \exp(-t)G(\xi)\,\rho_t \qquad \Longrightarrow \qquad \rho_t = \frac{-\rho}{1-\exp(-t)G(\xi)}, \tag{6.3}$$

$$\rho_x = \exp(-t)G(\xi)\,(1+\rho_x) \qquad \Longrightarrow \qquad \rho_x = \frac{\exp(-t)G(\xi)}{1-\exp(-t)G(\xi)}, \tag{6.4}$$

where $\xi = x + \rho$ and $G(\xi) = \dfrac{\mathrm{d}F}{\mathrm{d}\xi}(\xi)$. Thus:

$$\rho_t + \rho\,\rho_x = \frac{-\rho + \rho\,\exp(-t)G(\xi)}{1-\exp(-t)G(\xi)} = -\rho. \tag{6.5}$$

This shows that equation (6.1) is satisfied.

B. The characteristic form for (6.1) is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \rho \quad \text{and} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho. \tag{6.6}$$

The equation for ρ here has the general solution

$$\rho = F(\xi)\exp(-t), \tag{6.7}$$

where ξ is a label for the characteristic, and F is some arbitrary function. Then the general solution for the equation for x in (6.6) can be written as $x = -\rho + G(\xi)$, where G is some arbitrary function. In particular, if we take $G = \xi$, it is clear that we have $\xi = x + \rho$. Substituting this into (6.7), we obtain (6.2).

THE END.