Answers to P-Set # 02, 18.300 MIT (Spring 2022)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

February 16, 2022

Contents

List of Figures

1 Cola07. Small vibrations of a string under tension in 2D

1.1 Statement: Small vibrations of a string under tension in 2D

Here you are asked to derive the *equation for the transversal vibrations of a thin elastic string under tension*, under the following hypotheses

1. The string is homogeneous, with a *mass density* (mass per unit length) ρ — ρ is a constant.
2. The motion is restricted to the x-y plane. At equilibrium the string is described by $y = 0$ and $0 \leq x \leq L =$ string length. The *tension* $T > 0$ is then *constant*. See remark **1.1**, #1–3.
3. The string has no bending strength.
4. The amplitude of the vibrations is very small compared with their wavelength, and any longitudinal motion can be neglected. Thus:
   — The string can be described in terms of a *deformation function* $u = u(x, t)$,
   such that the equation for the *curve describing the string* is[1] $y = u(x, t)$.
   — The tension remains constant throughout — see remark **1.1**, #4.

Use the conservation of the transversal momentum to derive an equation for u. Thus:

(i) Obtain a formula for the transversal momentum density along the string (momentum per unit length).

(ii) Obtain a formula for the transversal momentum flux along the string (momentum per unit time).

(iii) Use the differential form of the conservation of transversal momentum to write a pde for u.

Hint: when doing (ii) remember that forces are momentum flux! Since there is no longitudinal motion, momentum can flow only via forces. Remember also that u_x is very small. Be careful with the signs here! ♣

The pde for u derived above applies for $0 < x < L$. For a solution to this equation to be determined, extra conditions on the solution u at each end are needed. These are called *boundary conditions*, and must be derived/modeled as well. Later on we will see that exactly one boundary condition (a restriction on u and its derivatives) is needed at each end, $x = 0$ and $x = L$.

Answer the following two extra questions:

(iv) What boundary condition applies at an end where **the string is tied?** (e.g., as in a guitar).

(v) What boundary condition applies at an end where **the string is free?**
For example, imagine a no-mass cart that can slide up and down (without friction) a vertical rod at $x = 0$, and tie the left end of the string to this cart. This keeps the left end of the string at $x = 0$, with the rod canceling the horizontal component of the tension force, but $u(0, t)$ can change.
A no-mass-no-friction cart is an idealization, approximating a setup where the mass and friction are small.

Remark 1.1 *Some details:*

#1 We idealize the string as a curve. This is justified as long as $\lambda \gg d$, where $\lambda =$ scale over which motion occurs, and $d =$ string diameter. This condition justifies item **3** above as well.

#2 The tension is generated by elastic forces (assume that the string is stretched). At a point along the string, the tension is the force with which each side (to the right or left of the point) pulls on the other side,[2] and it is directed along the direction tangent to the string.[3]

#3 At equilibrium the tension must be constant. For imagine that the tension is different at two points $a < b$ along the string. Then the segment $a \leq x \leq b$ would receive a net horizontal force (the difference in the tension values), and thus could not be at equilibrium.

#4 The hypothesis made imply that any changes in the string length can be neglected. Thus the amount of stretching, which generates the tension, does not change significantly anywhere. Hence the tension remains, essentially, equal to the equilibrium tension T. ♣

[1] The x-coordinate of any mass point on the string does not change in time.
[2] If you were to cut the string, this would be the force needed to keep the lips of the cut from separating.
[3] There are no normal forces, see item **3.**

1.2 Answer: Small vibrations of a string under tension in 2D

We have

(i) The transversal momentum per unit length along the string is given by $\boldsymbol{\rho_{\mathrm{mom}} = \rho\, u_t}$.

(ii) Momentum flux along the string happens because of transversal forces.
That is: the force in the y-direction that, at every point, one side of the string applies on the other side. This is just the y-component of the tension, which is given by $T\,\sin\theta$, where θ is the angle that the string tangent makes with the x-axis. However, since
u_x is small, $\theta \approx u_x$; the transversal momentum flux is given by $\boldsymbol{q_{\mathrm{mom}} = -T\, u_x}$.
Note about the sign here: when $u_x < 0$, the left side of the string pulls the right side up, thus generating a positive momentum flux. Vice-versa, when $u_x > 0$, momentum flows from right to left (negative). Thus the sign is as above. Note also that the x-component of the tension (horizontal force along the string) is $T\,\cos\theta = T$, since u_x is small. Thus it is constant, consistent with the approximation of no motion in the x direction.

(iii) From conservation $(\rho_{\mathrm{mom}})_t + (q_{\mathrm{mom}})_x = 0$. That is
$$\boldsymbol{\rho\, u_{tt} - T\, u_{xx} = 0}. \tag{1.1}$$
Alternatively
$$\boldsymbol{u_{tt} - c^2\, u_{xx} = 0}, \tag{1.2}$$
where $\boldsymbol{c^2 = T/\rho}$ is a speed (the wave speed).

(iv) **Tied end.** Suppose that the left end of the string is tied. Then its position is prescribed there (say: $y = a$ at $x = 0$), which leads to $u(0,\, t) = a$,
where a is a constant. More generally, we can have $u(0,\, t) = \sigma(t)$.

(v) **Free end.** Suppose the string's left end is free, so that there is no force there to balance any transversal component of the tension. Hence $u_x(0,\, t) = 0$.

In the problem statement a no-mass, no-friction, cart on a rod is mentioned as a way to achieve this. But an actual cart would have some mass, say M, and there would be friction (say, a force proportional to the cart velocity, opposing its motion). Then, since the cart position is given by $y = u(0,\, t)$, we can write
$$M\, u_{tt}(0,\, t) = -\nu\, u_t(0,\, t) + T\, u_x(0,\, t). \tag{1.3}$$
If τ is a typical time scale for the vibrations of the string, and λ is a typical wavelength, then $u_x(0,\, t) = 0$ *is a good approximation to this provided that* $\frac{T}{\lambda} \gg \frac{M}{\tau^2}$ and $\frac{T}{\lambda} \gg \frac{\nu}{\tau}$.
That is, provided that the two a-dimensional numbers $\boldsymbol{\dfrac{M\,\lambda}{T\,\tau^2}}$ and $\boldsymbol{\dfrac{\nu\,\lambda}{T\,\tau}}$ are small.

1.2.1 Conservation of energy

In the derivation above we obtained an equation for the string using, solely, the conservation of the transversal momentum. What happens with the other quantities that are conserved? They should be conserved. Let us check that this is so.

Conservation of mass. This is guaranteed by the parameterization we use. That is, each point $(x,\, u)$ follows a specific bit of the spring, with mass $\rho\,dx$. Thus mass is automatically conserved. This is typical of solid mechanics problems, where the parameterization tracks mass points in the object. In fluids, where individual particles are not tracked, and instead the flow velocity at each point in space is prescribed, mass conservation is not automatic, and must be enforced. On the other hand, in fluids the equations involve only first derivatives in time of the densities — while solids second order equations occur. This is because the velocities are variables in fluid problems, while in solids they result from the time derivatives of the displacements.

Conservation of longitudinal momentum. Since there is no motion in the x-direction, and the horizontal component of the tension forces is constant (see the end of item (ii) of the answer), the longitudinal momentum is conserved as well (trivially so). In fact, *the statement that the derivation above does not use the conservation of the longitudinal momentum is* <u>*incorrect*</u>*!* **It is used when concluding that the tension on the string is constant.**

Conservation of energy. To show that energy is conserved, we first need to calculate its density and its flux. The energy density has two components: **kinetic energy per unit length** $= \frac{1}{2}\,\rho\,u_t^2$, and the potential energy stored in the stretching of the string, which we need to compute. In fact we only need to compute the difference between the potential energy at any time, and some constant energy — specifically: the potential energy of the string at equilibrium. But this is easy, for then the tension is constant through the whole stretching process, so that the energy is the product of T times the change in length. This yields **potential energy per unit length** $= T\,\left(\sqrt{1+u_x^2}-1\right) = \frac{1}{2}\,T\,u_x^2$, since the arc-length is $\mathrm{d}s = \sqrt{1+u_x^2}\,\mathrm{d}x$, and u_x is small. Finally, the flux of energy is given by the work (per unit time) done by the transversal force (which we already calculated in (ii) of the answer). That is: **energy flux** $= -T\,u_x\,u_t$. It follows that the equation for the conservation of energy is

$$\left(\frac{1}{2}\,\rho\,u_t^2 + \frac{1}{2}\,T\,u_x^2\right)_t - (T\,u_x\,u_t)_x = 0. \tag{1.4}$$

It is easy to see that (1.1) guarantees this.
This calculation does not involve an internal energy, because in this model there is no mechanism for energy exchanges between mechanical and internal. This is the reason why energy conservation is "automatic".

2 DiAn01. Diffusion speed for ν constant (at what rate does an ink blob grow?)

2.1 Statement: Diffusion speed for ν constant

In the lectures it was shown that if $\Theta = \Theta(\vec{x},\,t)$ denotes the concentration of salt in water (e.g.: grams per liter) then, **assuming that there is no motion** by the water

$$\Theta_t = \nu\,\Delta\,\Theta, \tag{2.1}$$

where ν is the *diffusion coefficient* — which we assume here to be constant[4] — and Δ is the *Laplace operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The **same equation** (with a different value of ν) applies if, for example, Θ denotes sugar concentration, or the concentration of some coloring (e.g. ink).

Under the conditions where (2.1) applies, imagine that you inject a very tiny blob of ink inside the liquid.[5] Then the size of the blob will start increasing with time (due to the ink's diffusion). The blob's edge will cease to be sharp as time goes on, but here we will simplify things and assume that it remains sharp enough during the course of the experiment.

Make the approximation that, at time $t = 0$ (when you start the experiment) the blob is just a point. Then, *using qualitative and physical arguments only* **(do not attempt to solve the equation)**, *answer these questions:*

1. What are the dimensions of ν? Explain!
2. Argue that the shape of the blob is a sphere for $t > 0$.
3. Find a formula for the radius of the blob $R = R(t)$ as a function of time. There is a numerical multiplicative constant α, as in $R = \alpha\,f(t)$, which you will not be able to determine (without solving the p.d.e.), but **you should be able to get $f(t)$.**

Typical values for diffusion coefficients, as in the examples above, are $\nu = \gamma\,10^{-5}\,\mathrm{cm}^2/\mathrm{sec}$, where γ ranges[6] between $1/2$ and 2. **Assume that $\alpha = \gamma = 1$.** Then

[4] Generally ν need not be a constant; it may depend (for example) on the temperature.
[5] Carefully, so as not to start motion. The ink density must match that of water, to avoid gravity induced motion.
[6] Larger molecules diffuse slower than smaller ones; thus $\gamma(\text{sugar}) < \gamma(\text{salt})$.
Further, γ grows with temperature, and may even depend on the concentration.

4. What is the radius of the blob when $t = 1$ minute?

5. At what time does $R = 5$ cm?

These numbers should give you an idea of how long it would take to sweeten a cup of coffee if you just deposited a lump of sugar in it, and did not stir the coffee.
Hint. *The radius $\boldsymbol{R}$ is a length; a function of time. However, the problem has only one dimensional parameter,*[7] $\boldsymbol{\nu}$*. How then do you get a length R from a time $\boldsymbol{t}$? There is only one way!*

2.2 Answer: Diffusion speed for $\boldsymbol{\nu}$ constant

1. The diffusion constant must convert the dimensions of $\Delta\Theta$ to the dimensions of Θ_t. Hence $\boldsymbol{\nu}$ has *dimensions of length square over time.*

2. The situation is invariant under rotations. Hence the only shape the blob is allowed is a sphere. Asymmetries produced by motion, temperature gradients, etc., will destroy this. As long as the initial blob is small, asymmetries in its shape will not matter much.

3. We must find a relationship between x and t. There is no length scale provided by the initial conditions, the only dimensional parameter in the problem is ν. Hence it must be that

$$R = \alpha\sqrt{\nu\, t}, \tag{2.2}$$

where α is a numerical constant (no dimensions).

4. For $\alpha = 1$ and $\nu = 10^{-5}\,\text{cm}^2/\text{sec}$, equation (2.2) yields $\boldsymbol{R} \approx$ **0.245 mm** after one minute.

5. From equation (2.2), the time at which $R = L$ is given by

$$t = \frac{L^2}{\alpha^2\,\nu}. \tag{2.3}$$

For $\alpha = 1$ and $\nu = 10^{-5}\,\text{cm}^2/\text{sec}$, this yields $t = 2.5 \times 10^6$ sec $= 694.44$ hours for $L = 5$ cm. This number is a bit shocking, however it assumes no motion at all. Even a tiny amount of motion can change things dramatically. Just to get an idea: imagine a pot of radius 5 cm, and imagine that the liquid is slowly rotating in it, with the velocity at half the radius (2.5 cm) being 0.01 mm/sec. Then, in the time period above, this is enough for the liquid to go around the pot roughly 160 times! Hence even this tiny the motion is "too fast" to be neglected in the calculation, and the number above may not apply.

What happens when you stir the pot is that you mix things up, and reduce the distance diffusion has to work over. For example, if the 5 cm get reduced to 1 mm by mixing, the time is reduced by a factor of 50^2!
In the case of the cup of coffee with a lump of sugar at the bottom, just the fact that the coffee is cooling faster at the top (in contact with air) induces some motion that cuts down the time considerably — though you would still have to wait quite a bit if you do nor stir.

3 ExID51. Directional derivatives

3.1 Statement: Directional derivatives

Calculate the derivatives indicated below

[7] Note that at time $t = 0$ the blob has no dimension; it is a point!

1. Express $\dfrac{du}{ds}$ — for $u = u(x(s),\, y(s))$ — in terms of x, y, and u, given that:
$x(s) = e^s$, $y(s) = s\, e^s$, and u satisfies $x\, u_x + (y + x)\, u_y = 2\, x + u$.

2. Express $\dfrac{du}{dt}$ — for $u = u\left(t^2,\, \ln(t),\, \dfrac{1}{t}\right)$ — in terms of the partial derivatives of $u = u(x,\, y,\, z)$ and t.

3. Let $u = e^x\, \sin(y)$, and let $(r,\, \theta)$ be the polar coordinate radius and angle. Calculate $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$ in terms of x and y.

3.2 Answer: Directional derivatives

We have:

1. $\dfrac{du}{ds} = \dfrac{dx}{ds}\, u_x + \dfrac{dy}{ds}\, u_y = x\, u_x + (x + y)\, u_y = 2\, x + u.$

2. $\dfrac{du}{dt} = 2\, t\, u_x + \dfrac{1}{t}\, u_y - \dfrac{1}{t^2}\, u_z$, where the partial derivatives are evaluated at $x = t^2$, $y = \ln(t)$, and $z = \dfrac{1}{t}$.

3. Since $x = r\, \cos\theta$ and $y = r\, \sin\theta$, we have:
$r\, u_r = r\, \cos\theta\, u_x + r\, \sin\theta\, u_y = x\, e^x\, \sin(y) + y\, e^x\, \cos(y)$, and
$u_\theta = -r\, \sin\theta\, u_x + r\, \cos\theta\, u_y = -y\, e^x\, \sin(y) + x\, e^x\, \cos(y)$.

---

4 ExID03. Solve the one directional wave equation

4.1 Statement: Solve the one directional wave equation

Consider the one directional wave equation for $u = u(x,\, t)$, where c is a constant:

$$u_t + c u_x = 0. \tag{4.1}$$

Introduce the new independent variables $\boldsymbol{\eta = x - c\, t}$ and $\boldsymbol{\xi = x + c\, t}$, and change variables to write the equation for $\boldsymbol{u}$ as a function of these new variables: $\boldsymbol{u = u(\eta,\, \xi)}$. Using the transformed form of the equation, integrate it to show that it must be

$$u = f(\eta), \tag{4.2}$$

for some arbitrary function f. Hence, the solutions to (4.1) must have the form $u = f(x - c\, t)$.

4.2 Answer: Solve the one directional wave equation

Write $u = u(\eta, \xi)$, where $\eta = x - c\, t$ and $\xi = x + c\, t$. Then the chain rule yields:

$$\begin{aligned} u_t &= c u_\xi - c u_\eta, \\ u_x &= u_\xi + u_\eta, \end{aligned}$$

Thus, in these coordinates equation (4.1) reduces to $\boldsymbol{u_\xi = 0}$. Integrating this equation with respect to ξ, yields $\boldsymbol{u = f(\eta)}$, where f is some arbitrary function. This is equation (4.2).

---

5 TFPa02. Near constant initial data (linearize)

5.1 Statement: Near constant initial data (linearize)

Let
$$\boldsymbol{q = \frac{4\,q_m}{\rho_j^2}\,\rho\,(\rho_j - \rho)}, \quad (5.1)$$
where $\boldsymbol{q_m}$ is the maximum flow rate and $\boldsymbol{\rho_j}$ is the jamming concentration.

Let the concentration at time $t = 0$ be specified by
$$\boldsymbol{\rho(x,\,0) = \rho_0 + \epsilon\,\rho_j\,f(x)}, \quad (5.2)$$
where $\boldsymbol{0 < \rho_0 < \rho_j}$ is some constant density, $\boldsymbol{0 < \epsilon < (\rho_j - \rho_0)/\rho_j}$ is a numerical constant,
and (for some constant distance $\boldsymbol{D > 0}$)
$$f(x) = \left\{ \begin{array}{ll} 0 & \text{for } |x| \geq D, \\ 1 - \dfrac{x^2}{D^2} & \text{for } |x| \leq D, \end{array} \right\}. \quad (5.3)$$

If $\boldsymbol{0 < \epsilon \ll 1}$ is small (e.g.: $\epsilon = 0.1$), **use linear theory to determine the concentration $\boldsymbol{\rho(x,\,t)}$ at a later time. In particular:** set $\boldsymbol{\rho_0 = 0.75\,\rho_j}$, and find the wave speed and draw in space–time (x–t plane) the characteristics marking the front and back of the disturbance. On the same graph, plot the trajectories of the vehicles (which move with the space average velocity u) whose initial positions are $x = 0$ and $x = 2\,D$.

Repeat the exercise for $\boldsymbol{\rho_0 = 0.25\,\rho_j}$ and $\boldsymbol{\rho_0 = 0.5\,\rho_j}$. What is un-usual about the last case?

For the purposes of the plotting, **assume space and time units where $\boldsymbol{D = 1}$ and $\boldsymbol{q_m = \rho_j}$ — questions: is this possible? What are the units?**

5.2 Answer: Near constant initial data (linearize)

Write $\boldsymbol{\rho = \rho_0 + \epsilon\,\rho_j\,\tilde{\rho}}$. Then, for $0 < \epsilon \ll 1$, linear theory yields

$$\tilde{\rho}_t + c_0\,\tilde{\rho}_x = 0, \quad \text{where } c_0 = \frac{dq}{d\rho}(\rho_0) = \frac{4\,q_m}{\rho_j^2}\,(\rho_j - 2\,\rho_0) \quad \text{and } \tilde{\rho}(x,\,0) = f(x). \quad (5.4)$$

Hence
$$\boldsymbol{\tilde{\rho} = f(x - c_0\,t)}, \quad \text{where } \boldsymbol{f} \text{ is as in (5.3).} \quad (5.5)$$
Further, recall that the vehicle speed, in this linear approximation, is constant and given by
$$u_0 = \frac{q(\rho_0)}{\rho_0} = \frac{4\,q_m}{\rho_j^2}\,(\rho_j - \rho_0). \quad (5.6)$$
It follows that:

1. For $\boldsymbol{\rho_0 = 0.75\,\rho_j}$, we have: $\boldsymbol{c_0 = -\dfrac{2\,q_m}{\rho_j}}$ and $\boldsymbol{u_0 = \dfrac{q_m}{\rho_j}}$. Left panel in figure 5.1.

2. For $\boldsymbol{\rho_0 = 0.25\,\rho_j}$, we have: $\boldsymbol{c_0 = \dfrac{2\,q_m}{\rho_j}}$ and $\boldsymbol{u_0 = 3\,\dfrac{q_m}{\rho_j}}$. Middle panel in figure 5.1.

3. For $\boldsymbol{\rho_0 = 0.50\,\rho_j}$, we have: $\boldsymbol{c_0 = 0}$ and $\boldsymbol{u_0 = 2\,\dfrac{q_m}{\rho_j}}$. Right panel in figure 5.1.

In each of the cases, the following applies to the Traffic Waves (TW): **1.** *TW move backwards relative to the road.* **2.** *TW move forwards relative to the road.* **3.** *TW do not move relative to the road.*
In all cases *the cars move faster than the waves.*

Finally, notice that units that yield $D = 1$ and $q_m = \rho_j$ are possible: take D as the unit of length, and $D\,\rho_j/q_m$ as the unit of time.

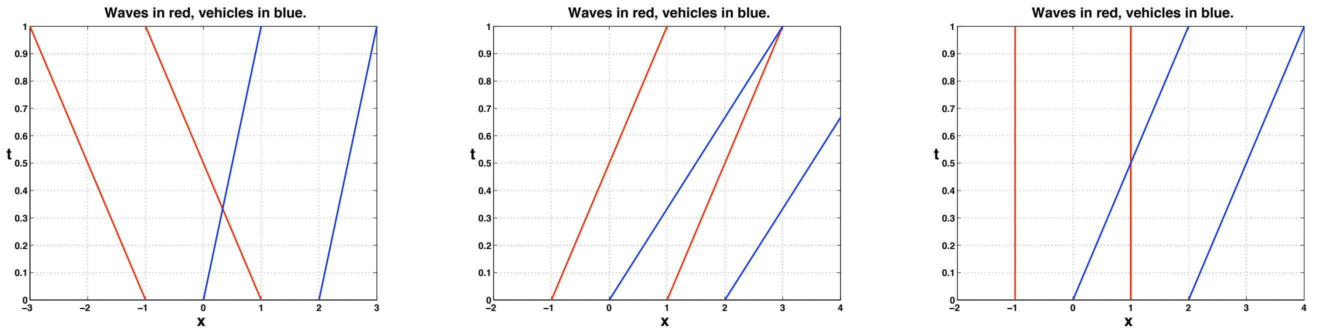

Figure 5.1: TFPa02. Characteristics (red) and vehicle paths (blue) for the cases **1–3** in the answer. The characteristics track the front and back of the traffic flow disturbance. The vehicle's tracked are those starting at $x = 0$ and $x = 2D = 2$.

6 TFPa06. Solve a first order linear pde

6.1 Statement: Solve a first order linear pde

If $a \neq 0$, b, and c are constants, show that any function $u = u(x,\, y)$
that satisfies the partial differential equation

$$a\, u_x + b\, u_y = c\, u, \tag{6.1}$$

must be of the form

$$u = e^{c\,x/a}\, F(b\,x - a\,y), \tag{6.2}$$

where F is an arbitrary function.
Hint: *Use* $\xi = b\,x - a\,y$ *and* $\eta = x$ *as the new independent variables.*

6.2 Answer: Solve a first order linear pde

As long as $a \neq 0$ we can change coordinates from $(x,\, y)$ to $(\xi,\, \eta)$, where $\xi = b\,x - a\,y$ and $\eta = x$. Then, using the chain rule, we have $u_x = b\,u_\xi + u_\eta$ and $u_y = -a\,u_\xi$.
Hence, in these new coordinates the equation becomes

$$a\, u_\eta = c\, u. \tag{6.3}$$

This is, basically, an ode in η, with general solution $u = F(\xi)\, \exp\,(c\,\eta/a)$ — where
F is the "constant" of integration, which can be different for different values of ξ. Thus equation (6.2) follows.

Note: The change of coordinates above is no good when $a = 0$ (why?), and neither is our solution above. However, when $a = 0 \neq b$, the equation
is just $b\,u_y = c\,u$, with solution

$$u = F(x)\, e^{c\,y/b}. \tag{6.4}$$

Finally, if $a = b = 0$, there is not much of an equation to solve.

7 TFPa07. Solve a first order linear pde

7.1 Statement: Solve a first order linear pde

If c and β are a constants, find the solution to

$$u_t + c\, u_x = -\beta^2\, u, \tag{7.1}$$

with initial condition $u(x,\, 0) = \sin x$.

7.2 Answer: Solve a first order linear pde

Introduce the new system of coordinates given by $\xi = x - c\,t$ and $\eta = t$. In these coordinates the equation becomes $\boldsymbol{u_\eta = -\beta^2\, u}$, which is very easy to solve.
We obtain the general solution

$$u = F(\xi)\, e^{-\beta^2\,\eta} = F(x - c\,t)\, e^{-\beta^2\, t}, \tag{7.2}$$

where F is an arbitrary function.
The solution satisfying $u(x,\,0) = \sin x$ is then

$$u = \sin(x - c\,t)\, e^{-\beta^2\, t}. \tag{7.3}$$

8 TFPa08. General solution 1-D wave equation

8.1 Statement: General solution 1-D wave equation

If $\xi = x - c\,t$ and $\eta = x + c\,t$, show that the wave equation

$$u_{tt} - c^2\, u_{xx} = 0 \tag{8.1}$$

transforms to $\boldsymbol{u_{\xi\,\eta} = 0}$. Then deduce that

$$u = F(x - c\,t) + G(x + c\,t) \tag{8.2}$$

is the general solution to (8.1), with F and G arbitrary functions.
Interpret this result in terms of traveling waves.
If $\boldsymbol{u(x,\,0) = \phi(x)}$ and $\boldsymbol{u_t(x,\,0) = \psi(x)}$,
show that

$$u = \frac{1}{2}\left(\phi(x - c\,t) + \phi(x + c\,t)\right) + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} \psi(\xi)\,\mathrm{d}\xi. \tag{8.3}$$

Given this formula, **describe (in words) what happens when $\psi = 0$.**

8.2 Answer: General solution 1-D wave equation

Substituting $\boldsymbol{u = u(x - c\,t,\; x + c\,t)}$ into (8.1) yields $\boldsymbol{4\,c^2\, u_{\xi,\eta} = 0}$. That is: $\boldsymbol{u_{\xi,\eta} = 0}$. Hence u_ξ is not a function of η, namely $u_\xi = f(\xi)$, for some function f. Integrating this yields $\boldsymbol{F(\xi) + G(\eta)}$, where $F' = f$, and G is the "constant" of integration (which may depend on η). Thus we obtain (8.2).

Equation (8.2) tells us that *the general solution to the wave equation in 1-D consists of the superposition of two traveling waves of arbitrary shapes, both moving at speed c, one to the left and the other to the right. The waves' shape do not change as the waves travel.*

Substituting the initial conditions $u(x,\,0) = \phi(x)$ and $u_t(x,\,0) = \psi(x)$ into the general solution (8.2) that we just obtained yields

$$F(x) + G(x) = \phi(x) \quad \text{and} \quad -\,c\,F'(x) + c\,G'(x) = \psi(x). \tag{8.4}$$

Hence, upon integration, we obtain

$$F(x) + G(x) = \phi(x) \;\; \text{and} \;\; -c\,F(x) + c\,G(x) = \int_0^x \psi(s)\,ds + a, \tag{8.5}$$

where a is a constant of integration.
It follows that

$$F(x) = \frac{1}{2}\,\phi(x) - \frac{1}{2\,c}\int_0^x \psi(s)\,ds - b \quad \text{and} \quad G(x) = \frac{1}{2}\,\phi(x) + \frac{1}{2\,c}\int_0^x \psi(s)\,ds + b, \tag{8.6}$$

where $b = a/(2\,c)$. Substituting this result into (8.2) gives (8.3).

Consider the case $\psi = 0$, so that **the solution starts from some given shape ϕ, with zero initial velocity.** Then (8.3) shows that *the shape splits into two equal pulses (equal in shape to the initial deformation, but of half the original amplitude), one moving to the right at speed c, and the other to the left at speed $-c$.*

THE END.