Answers to P-Set # 01, 18.300 MIT (Spring 2022)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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Contents

List of Figures

1 Compute a channel flow rate function #01

1.1 Statement: Compute a channel flow rate function #01

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{1.1}$$

where $A = A(x,t)$ is the cross-sectional filled area of the river bed, x measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate q should be a function of A only [1] follows from the assumption of quasi-equilibrium. Then q is determined by a local balance between the friction forces and the force of gravity down the river bed.

Assume now a man-made channel, with uniform triangular cross-section† and a uniform (small) downward slope, characterized by an angle θ. Assume also that the frictional forces are proportional to the product of the flow velocity u down the channel, and the wetted perimeter P_w of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function Q should have.** † Isosceles triangle, with bottom angle ϕ.

Hints: (1) $Q = u\,A$, *where* u *is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter* P_w *is proportional to some power of* A*.*

1.2 Answer: Compute a channel flow rate function #01

The wetted perimeter is proportional to the square root of the filled cross-sectional area. That is: [2] $P_w = C_w \sqrt{A}$, where $\boldsymbol{C_w = \sqrt{8/\sin(\phi)}}$ and ϕ is the *bottom angle of the triangular channel bed.* Thus the frictional forces (per unit length) along the river bed are given by $F_f = C_f\, P_w\, u = C_f\, C_w\, u \sqrt{A}$ — where u is the flow velocity and C_f is a *friction coefficient.*

On the other hand, the component of the force of gravity (per unit length) along the channel bed is given by $F_g = \rho\, g\, \sin(\theta)\, A$ — where θ is the *angle that the channel bed makes with the horizontal,* g is the *acceleration of gravity,* and ρ is the *density of the water* in the channel.

[1] Possibly also x. That is: $q = Q(x,\, A)$, to account for non-uniformities along the river.
[2] Write the height and base of the water-filled-triangle in terms of the half bottom angle and the wetted side $= P_w/2$.

From the quasi-equilibrium assumption $F_f = F_g$. This yields $u = \dfrac{\rho\, g\, \sin(\theta)}{C_f\, C_w}\sqrt{A}$. Hence, since $q = u\, A$, it follows that:

$$\boldsymbol{Q = \frac{\rho\, g\, \sin(\theta)}{C_f\, C_w}\, A^{\frac{3}{2}}.} \tag{1.2}$$

Note that q is a convex function of A.

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2 Conservation of probability in QM

2.1 Conservation of probability in QM

In non-relativistic quantum mechanics the motion of a point particle in a potential V is described by Schrödinger's equation.

$$i\,\hbar\,\psi_t = -\frac{\hbar^2}{2\,m}\,\psi_{xx} + V(x)\,\psi \quad \text{in 1D,} \tag{2.1}$$

where $\hbar = \frac{h}{2\,\pi}$ is the Plank constant divided by $2\,\pi$,
$\psi = \psi(x,\,t)$ is the (*complex valued*) wave function,
m the particle's mass, and i is the imaginary unit.
The interpretation is that [3]

$$\tilde{\rho} = |\psi|^2 = \psi\,\psi^* \tag{2.2}$$

is the pdf [probability distribution function] (pdf)
for the particle position. That is, the probability
of finding the particle in any interval $a < x < b$ is [4]

$$\int_a^b \tilde{\rho}\, dx. \tag{2.3}$$

Now: probability is conserved, and $\tilde{\rho}$ is its density. **Question: What is the probability flux?**
Hint. Use (2.1) to find an equation of the form $\tilde{\rho}_t + \tilde{q}_x = 0$. The flux is then $\tilde{q}$.
Warning: check that the flux you obtain is real valued.

2.2 Answer: Conservation of probability in QM

Multiply (2.1) by ψ^*. Take the resulting equation, and subtract from it its complex conjugate. After a bit of manipulation, the result can be written in the form

$$i\,\hbar\,\tilde{\rho}_t + \frac{\hbar^2}{2\,m}\,(\psi_x\,\psi^* - \psi\,\psi_x^*)_x = 0. \tag{2.4}$$

Thus the probability flux is

$$\tilde{q} = \frac{\hbar}{2\,i\,m}\,(\psi_x\,\psi^* - \psi\,\psi_x^*), \tag{2.5}$$

which is *real valued*, as expected.

Alternatively, write the wave function using polar variables $\psi = r\,e^{i\,\theta}$, where $\tilde{\rho} = r^2$. Substituting this into (2.1), multiplying by $e^{-i\,\theta}$, and taking real and imaginary parts, leads to the two equations

$$\hbar\, r_t = -\frac{\hbar^2}{2\,m}\,(2\,r_x\,\theta_x + r\,\theta_{xx})\,, \tag{2.6}$$

$$\hbar\, r\,\theta_t = \frac{\hbar^2}{2\,m}\,\left(r_{xx} - r\,\theta_x^2\right) - V\,r. \tag{2.7}$$

[3] Here $*$ indicates the complex conjugate.
[4] ψ should be normalized so that $\int \tilde{\rho}\, dx = 1$, where the integral is over the whole domain where the particle resides. The units for ψ are $1/\sqrt{\text{length}}$ in 1D.

Multiplying (2.6) by $\frac{2}{\hbar}\,r$ yields

$$\tilde{\rho}_t + \left(\frac{\hbar}{m}\,\tilde{\rho}\,\theta_x\right)_x = 0. \tag{2.8}$$

This corresponds to

$$\tilde{q} = \frac{\hbar}{m}\,\rho\,\theta_x \tag{2.9}$$

— which, you can check, **is the same as (2.5).**

2.2.1 Fluid analogy and Madelung

One can associate a flow velocity to a conserved quantity by writing the flux as the density times the velocity (this defines the velocity). In the current example, it follows from (2.9) that

$$u = \frac{\hbar}{m}\,\theta_x \quad \text{is the probability density flow velocity.} \tag{2.10}$$

Introduce now $\boldsymbol{\rho = m\,\tilde{\rho}}$, so that ρ is the "mass probability distribution function" (with mass per unit length units). Then rewrite (2.6–2.7) in the form (see "Details" below)

$$\begin{aligned} \rho_t + (\rho\,u)_x &= 0, & (2.11)\\ (\rho\,u)_t + (\rho\,u^2 + p)_x &= -\tilde{\rho}\,V_x, & (2.12) \end{aligned}$$

where $p = \dfrac{\hbar^2}{2\,m^2}\left(R_x^2 - R\,R_{xx}\right) = -\dfrac{\hbar^2}{2\,m^2}\,(\ln R)_{xx} = -\dfrac{\hbar^2}{4\,m^2}\,(\ln\rho)_{xx}$ and $R = \sqrt{\rho}$.

Except for the strange form of the pressure, this is the same as the isentropic Euler equations of Gas Dynamics, with a probability-weighted body force $F = -\tilde{\rho}\,V_x$.

1–This analogy was noted by E. Madelung: Quantentheorie in Hydrodynamischen form. Z. Phys. **40**:322-326 (1926).

2–In classical mechanics the force $-V_x$ is applied to the particle, at the particle location. Here the force is applied to the whole mass pdf, weighted by the position pdf.

Note also that, according to QM, the expected value of the particle momentum is $-i\,\hbar\int\psi^*\,\psi_x\,dx$. This also has a fluid dynamic analog, since $-i\,\hbar\int\psi^*\,\psi_x\,dx = \int\rho\,u\,dx =$ total momentum.

Details: (2.11) is the same as (2.8). To obtain (2.12), multiply (2.7) by $\frac{1}{m\,r}$, and next differentiate with respect to x. This leads to

$$u_t + u\,u_x - \frac{\hbar^2}{2\,m^2}\left(\frac{r_{xx}}{r}\right)_x = -\frac{1}{m}\,V_x.$$

Then multiply by ρ. Upon use of (2.11), it is easy to see that this yields (2.12).

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3 Dispersive Waves and Modulation

3.1 Statement: Dispersive Waves and Modulation

Consider the following linear partial differential equations for the scalar function $\boldsymbol{u = u(x,\,t)}$:

$$\begin{aligned} \boldsymbol{u_t + c\,u_x + d\,u_{xxx}} &= \boldsymbol{0}, & (3.1)\\ \boldsymbol{u_{tt} - u_{xx} + a\,u} &= \boldsymbol{0}, & (3.2)\\ \boldsymbol{i\,u_t + b\,u + g\,u_{xx}} &= \boldsymbol{0}, & (3.3) \end{aligned}$$

where the equations are written in a-dimensional variables, $\boldsymbol{(c,\,d,\,a,\,b,\,g)}$ are real constants, and $\boldsymbol{a > 0}$. These equations arise in many applications, but we will not be concerned with them here. It should be clear that, in all three cases,

$$\boldsymbol{u = A\,e^{i\,(k\,x - \omega\,t + \theta_0)}}, \quad \text{where} \quad \boldsymbol{\omega = \Omega(k)}, \tag{3.4}$$

is a solution of the equations, for any real constants $A > 0$, θ_0, k, and ω, provided that

M1. For equation (3.1): $\Omega(k) = c\,k - d\,k^3$ .... **Verify that this is true.**

M2. For equation (3.2): $\Omega(k) = \pm\sqrt{a + k^2}$ .... **Verify that this is true.**

M3. For equation (3.3): $\Omega(k) = -b + g\,k^2$ .... **Verify that this is true.**

Note that the general solution to the equations can be written as a linear combination of solutions of this type, via **Fourier Series** and **Fourier Transforms** — we will see this later in the semester.

Remark 3.1 *Solutions such as that in (3.4) represent monochromatic sinusoidal traveling waves, with* **amplitude** A, **phase** $\theta = k\,x - \omega\,t + \theta_0$, **wave number** k, *and* **angular frequency** ω. *The* **wave length** *and* **wave period** *are* $\lambda = 2\,\pi/k$ *and* $\tau = 2\,\pi/\omega$, *respectively. The wave profile's crests and troughs move at the speed given by* $\theta =$ *constant, namely: the* **phase speed** $c_p = \omega/k$. ♣

Remark 3.2 *In all three cases,* $\Omega = \Omega(k)$ **is a real valued function of** k**, with** $\frac{d^2\Omega}{dk^2} \neq 0$ — *i.e.:* Ω *is* **not** *a linear function of* k. *Because of this, we say that the* **equations are dispersive** *and call* Ω **the dispersion function.** *The (non-constant) velocity* $c_g = c_g(k) = d\omega/dk$ *is called the* **group speed**, *and the objective of this problem is to find out what the meaning of* c_g *is.*

The reason for the name "dispersive" is as follows: In a dispersive system, waves with different wavelengths propagate at different speeds. Thus, a localized initial disturbance, made up of many modes of different wavelengths, will **disperse** *in time, as the waves cease to add up in the proper phases to guarantee a localized solution. This is because localization depends on destructive interference, outside some small region, of all the modes* $a(k)\,e^{i\,(k\,x+\theta_0)}$ *making up the initial disturbance. However, since these modes propagate at different speeds, the phase coherence needed for destructive interference is destroyed by the time evolution.* **This phenomena is illustrated in figure** 3.1. ♣

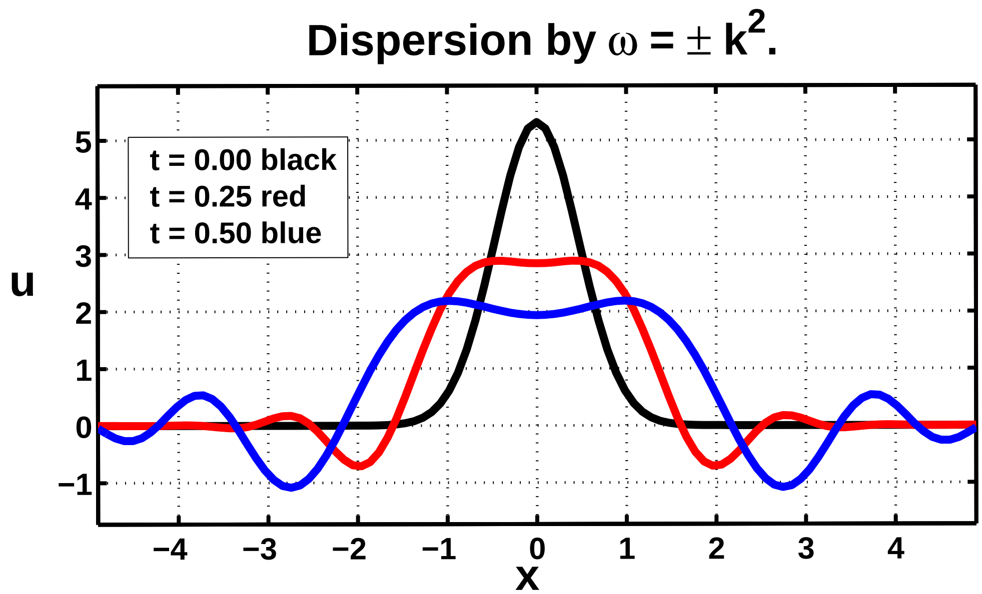

Figure 3.1: Example of dispersion: initial "Gaussian" bump, as it evolves under a dispersive equation with $\Omega(k) = \pm k^2$ — i.e.: $u_{tt} + u_{xxxx} = 0$. The solution at times $t = 0, 1/4, 1/2$ displayed. As the initial lump's phase coherence is destroyed by dispersion, localization is lost, and the bump "disperses". The solution $u = u(x,\,t)$ is given by the Fourier integral $u = \text{Re}\left(\int_{-\infty}^{\infty} a(k)e^{i(k\,x - k^2\,t)}\text{d}k\right)$, where $a(k) = e^{-k^2/9}$.

The tasks to be performed

TASK 1. verify **M1** through **M3**, above below equation (3.4).

TASK 2: Consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number k and angular frequency ω are related by a dispersion relation

$$\omega = \Omega(k). \tag{3.5}$$

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a *local wave number* $\boldsymbol{k = k(x,\, t)}$ and a *local angular frequency* $\boldsymbol{\omega = \omega(x,\, t)}$. In particular, assume that both k and ω vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods — *on the other hand, they may change considerably over many wave lengths or wave periods.* Then

Assuming conservation of wave crests, derive equations governing k and ω.

These equations are called the **Wave Modulation Equations.**

Remark 3.3 *The assumption that k and ω vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.* ♣

Remark 3.4 **Why is it reasonable to assume that the wave crests are conserved?** *The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where the assumption of slow variation is invalid. There are some pretty interesting research problems in pattern formation that are related to this point.* ♣

Hint 3.1 *It should be clear that one of the equations is $\omega = \Omega(k)$, since the solution behaves locally like a monochromatic wave* (this is the "quasi-equilibrium" approximation in this context). *For the second equation, express the density of wave crests (and its flux) in terms of k and ω. To figure this out, think of the following questions* (i) How many wave crests are there per unit length for a sinusoidal wave? (ii) How many wave crests pass through a fixed point in space, per unit time, for a sinusoidal wave? *Then write the equation for the conservation of wave crests using these quantities.* ♣

3.2 Answer: Dispersive Waves and Modulation

The wavelength is related to the wave number by $\boldsymbol{\lambda = \lambda(x,\, t) = 2\pi/k}$, while for the wave period τ we have $\boldsymbol{\tau = \tau(x,\, t) = 2\pi/\omega}$. Thus:

M4. $\frac{1}{\lambda} = \frac{1}{2\pi}\, k$ is the number of waves per unit length $=$ **wave crest density.**

M5. $\frac{1}{\tau} = \frac{1}{2\pi}\, \omega$ is the number of waves per unit time $=$ **wave crest flux.**

Hence, if the waves are conserved, we can write the **conservation of wave crests equation:**

$$\boldsymbol{k_t + \omega_x = 0,} \tag{3.6}$$

where $\boldsymbol{\omega = \Omega(k)}$, and we have eliminated the common $\boldsymbol{2\pi}$ factor.

Remark 3.5 *The wave's crests move at the* **phase speed:** $\boldsymbol{c_p = c_p(k) = \omega/k = \Omega(k)/k}$.
This is easy to see by noticing that we can write
$\exp\{i\,(k\,x - \omega\,t)\} = \exp\{i\,k\,(x - c_p\,t)\}$ *in (3.4). Since $\boldsymbol{\omega = c_p\, k}$,* this velocity plays in this theory the same role as the car speed in Traffic Flow. ♣

Remark 3.6 *Substituting $\omega = \Omega(k)$ into equation (3.6), and using the chain rule, we obtain:*

$$k_t + c_g(k)\, k_x = 0, \tag{3.7}$$

where $c_g = d\Omega/dk$ is the **group speed**. *Clearly,* $\boldsymbol{c_g}$ plays the same role here as the characteristic wave speed does in Traffic Flow. *Changes in k and ω travel at this speed. In particular, a "wave package" travels at this speed. It can also be shown that* $\boldsymbol{c_g}$ is the speed at which the wave energy propagates. ♣

Remark 3.7 *You may have encountered the notion of group speed in earlier courses, when studying the* beating *phenomena. Namely, adding two sinusoidal waves of close frequency and wavelength:*

$$\Phi = cos(k_1\, x - \omega_1\, t) + cos(k_2\, x - \omega_2\, t), \tag{3.8}$$

where $k_1 \approx k_2$ and $\omega_1 \approx \omega_2$, produces "beats" propagating at a velocity $\displaystyle \frac{\boldsymbol{\Delta\omega}}{\boldsymbol{\Delta k}} \approx \boldsymbol{c_g} =$ **group speed.**
This follows because we can write

$$\begin{aligned} \Phi &= 2\cos\left(\frac{k_1 - k_2}{2}\, x - \frac{\omega_1 - \omega_2}{2}\, t\right)\cos\left(\frac{k_1 + k_2}{2}\, x - \frac{\omega_1 + \omega_2}{2}\, t\right) \\ &= 2\cos\left(\frac{\Delta k}{2}\, x - \frac{\Delta\omega}{2}\, t\right)\cos\left(\bar{k}\, x - \bar{\omega}\, t\right), \end{aligned} \tag{3.9}$$

where $\bar{k}$ and $\bar{\omega}$ are the average wave number and angular frequency. This is the simplest example of a slowly modulated wave: The amplitude modulation is provided by the $2\cos((\Delta k\, x - \Delta\omega\, t)/2)$ factor, while the wave number $\bar{k}$, and wave frequency $\bar{\omega}$ remain constant. ♣

Remark 3.8 Where is the fact that $\boldsymbol{k}$ and $\boldsymbol{\omega}$ vary "slowly" in space and time used?

1. *So that we can talk about a wavelength and a frequency for the waves. For these things to make sense the wave must (locally) look like a plane wave — with a* $\boldsymbol{k}$ *and a* $\boldsymbol{\omega}$ *that are (essentially) constant over a few wavelengths or wave periods.*
2. *To write $\omega = \Omega(k)$ for the variables k and ω. This is the relationship that applies to plane, monochromatic waves — where k and ω are constants. Thus, to be (approximately) valid in the variable case, the changes in k and ω must be small over a few wave periods or wavelengths.* ♣

4 Fundamental Diagram of Traffic Flow #01

4.1 Statement: Fundamental Diagram of Traffic Flow #01

The desired car velocity $u = U(\rho)$ has its maximum, u_m, at $\rho = 0$, and vanishes at the jamming density, ρ_J. Assuming that U is a linear function of ρ, write a formula for the flow rate $q = Q(\rho)$. What is the road capacity q_m? What is the wave velocity $c = c(\rho)$?

4.2 Answer: Fundamental Diagram of Traffic Flow #01

From the stated assumptions

$$U = u_m\left(1 - \frac{\rho}{\rho_J}\right). \tag{4.1}$$

Hence, from $q = \rho\, u$, it follows that

$$Q = u_m\, \rho\left(1 - \frac{\rho}{\rho_J}\right) \quad \Longrightarrow \quad q_m = \frac{1}{2}\, \rho_J\, u_m. \tag{4.2}$$

Furthermore

$$c = \frac{dq}{d\rho} = u_m\left(1 - 2\,\frac{\rho}{\rho_J}\right). \tag{4.3}$$

5 Fundamental Diagram of Traffic Flow #03

5.1 Statement: Fundamental Diagram of Traffic Flow #03

Many state laws state that: for each 10 mph (16 kph) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 ft (5 m)). There is another law that gives a maximum speed limit (assume that this is 50 mph (80 kph)). **Find the flow of cars as a function of density, $q = q(\rho)$**, that results from these two laws.

The state laws on following distances stated in the prior paragraph were developed in order to prescribe a spacing between cars such that rear-end collisions could be avoided, as follows:

a. Assume that a car stops instantaneously. *How far would the car following it travel* if moving at u mph and
  a1. The driver's reaction time is τ, and
  a2. After a delay τ, the car slows down at a constant maximum deceleration α.

b. The calculation in part **a** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate α, but the driver in the following car still takes a time τ to react. *How far back does a car have to be, traveling at u mph, in order to prevent a rear-end collision?*

c. Show that the law described in the first paragraph of this problem corresponds to part **b**, if the human reaction time is about 1 sec. and the length of a car is about 16 ft (5 m).

Note: *What part **c** is asking you to do is to justify/derive the state law prescription, using the calculations in part **b** to arrive at the minimum car-to-car separation needed to avoid a collision when the cars are forced to brake.*

5.2 Answer: Fundamental Diagram of Traffic Flow #03

Assume that the drivers follow the state law prescription exactly. Then [5]

$$d = \frac{L\,u}{V}, \tag{5.1}$$

where d is the distance to the next car, L is the car length, u is the car velocity and V is the law "trigger" velocity, as in:

State Law: *Maintain a distance of one car length for each V increase in velocity.*

Typical numbers are $V = 10$ mph $= 16$ kph and $L = 16$ ft $= 5$ m.

Since we then end up with one car for every $d + L$ distance, equation (5.1) above leads to the velocity–density relationship

$$\frac{1}{\rho} = L + d = L\left(1 + \frac{u}{V}\right) \quad \text{or} \quad u = \left(\frac{1}{\rho\,L} - 1\right) V. \tag{5.2}$$

Thus the car flux is given by

$$q = u\,\rho = \left(\frac{1}{L} - \rho\right) V \implies c = \frac{dq}{d\rho} = -V. \tag{5.3}$$

This gives a constant wave speed c, and a car velocity that goes to infinity as the density vanishes. This because we have not yet enforced the **speed limit,** which yields

$$u = \min\left\{u_m,\ \left(\frac{1}{\rho\,L} - 1\right) V\right\} \quad \text{and} \quad q = \min\left\{\rho\,u_m,\ \left(\frac{1}{L} - \rho\right) V\right\}, \tag{5.4}$$

where $u_m =$ speed limit. The critical density below which $u = u_m$, is

$$\rho_c = \frac{V}{L\,(V + u_M)}. \tag{5.5}$$

Then (5.2 – 5.3) applies for $\rho \geq \rho_c$. Then, if $u < u_m$ we can recover $\rho > \rho_c$ from (5.2 – 5.3). On the other hand, $u = u_m$ for any $\rho \leq \rho_c$.

Next we motivate the state laws by a simple calculation involving two facts: (i) Drivers have a finite reaction time, $\tau > 0$. (ii) Cars do not change velocity instantaneously, but do so with a finite deceleration $\alpha > 0$. To simplify matters, here we assume that τ is the same for all the drivers, and that α is constant (and the same for all cars).

[5] To understand this note that d/L is the number of car lengths of separation, while u/V is the number of "trigger" velocities.

Imagine now two cars, one behind the other, traveling at the same (constant) speed u. At some point, the car ahead (car #1) starts braking. It then travels a distance

$$D_1 = \frac{1}{2}\frac{u^2}{\alpha} \tag{5.6}$$

from the moment the brakes are applied to the moment it stops. On the other hand, the distance traveled by the car behind (car #2) from the moment the driver sees that he must stop [6] till the car actually stops, includes the driver's reaction time. That is

$$D_2 = u\,\tau + \frac{1}{2}\frac{u^2}{\alpha}. \tag{5.7}$$

It then follows that, *in order to avoid a rear end collision, the distance between two cars traveling at speed **u** must be at least* $u\,\tau$. This yields the following formula for the "trigger" velocity

$$V = \frac{L}{\tau}. \tag{5.8}$$

In particular, $L = 16$ ft and $\tau = 1$ s yields $V = 16$ ft/s $= 10.9$ mph — note that 1 mile $= 5280$ ft and 1 hr $= 3600$ s. This trigger velocity is pretty close to the one used in many state laws.

Note: The calculation above is a bit sloppy, for it only checks that car #2 is still behind car #1 once they stop. What one should check is that car #2 stays behind car #1 at *all* times. However car #2 is always moving at a speed equal to or greater than that of car #1. This because it starts slowing down, at the same rate and starting from the same speed, later. Thus the distance between the two cars is a non-increasing function of time. It follows that it is enough to check that it is positive once they stop, to know that it was always positive.

6 ExID03. Single variable implicit differentiation

6.1 Statement: Single variable implicit differentiation

In each case compute $y' = \dfrac{dy}{dp}$ as a function of y and p, given that $y = y(p)$ satisfies:

1. $p^3 + p\,y + 2 = 0$.
2. $y = \sin(y + p)$.
3. $\ln(y) = p$.
4. $\cos^2(y) = p$, for $p > 0$.
5. $y = f(c - y\,p)$.
6. $y = f(p - c\,y)$.

Note: in (5) and (6) f is an arbitrary function, and c is a constant.

6.2 Answer: Single variable implicit differentiation

1. $p^3 + py + 2 = 0$ implies: $3p^2 + y + p\,\dfrac{dy}{dp} = 0$. Thus $\dfrac{dy}{dp} = -\dfrac{3p^2 + y}{p}$.

2. $y = \sin(y + p)$ implies: $\dfrac{dy}{dp} = \left(\dfrac{dy}{dp} + 1\right)\cos(y + p)$. Thus $\dfrac{dy}{dp} = \dfrac{\cos(y + p)}{1 - \cos(y + p)}$.

3. $\ln(y) = p$ implies: $\dfrac{dy}{dp}\dfrac{1}{y} = 1$. Thus $\dfrac{dy}{dp} = y$.

4. $\cos^2(y) = p$, for $p > 0$ implies: $-2\cos y\,\sin y\,\dfrac{dy}{dp} = 1$. Thus $\dfrac{dy}{dp} = -\dfrac{1}{\sin(2\,y)}$.

[6] Say, the brake lights for the car ahead turn on.

5. $y = f(c - y\,p)$, where f is an arbitrary function and c is a constant implies:
$$\frac{dy}{dp} = -f'(c - y\,p)\left(p\,\frac{dy}{dp} + y\right). \quad \text{Thus} \quad \frac{dy}{dp} = -\frac{y\,f'(c - y\,p)}{1 + p\,f'(c - y\,p)}.$$
6. $y = f(p - c\,y)$, where f is an arbitrary function and c is a constant implies:
$$\frac{dy}{dp} = f'(p - c\,y)\left(1 - c\,\frac{dy}{dp}\right). \quad \text{Thus} \quad \frac{dy}{dp} = \frac{f'(p - c\,y)}{1 + c\,f'(p - c\,y)}.$$

7 ExID14. Two variable implicit differentiation

7.1 Statement: Two variable implicit differentiation

In each case compute $u_x = \dfrac{\partial u}{\partial x}$ and $u_p = \dfrac{\partial u}{\partial p}$ (as functions of u, x, and p), given that $u = u(x,\,p)$ satisfies:

1. $\cos(p^2\,u) = p\,e^{-x^2}$. 2. $p = \cos(x + u)$. 3. $u = p\,f(x + u)$.

Note: In (3) f is an arbitrary function of a single variable, $f = f(\zeta)$.

7.2 Answer: Two variable implicit differentiation

1. Upon taking partial derivatives with respect to x and p, $\cos(p^2\,u) = p\,e^{-x^2}$ yields:
$$p^2\,u_x \sin(p^2\,u) = 2\,x\,p\,e^{-x^2} \quad \text{and} \quad (2\,p\,u + p^2\,u_p)\,\sin(p^2\,u) = -e^{-x^2}.$$
$$\text{Thus:} \quad u_x = \frac{2\,x\,p\,e^{-x^2}}{p^2\,\sin(p^2\,u)} \quad \text{and} \quad u_p = -\frac{2\,p\,u\,\sin(p^2\,u) + e^{-x^2}}{p^2\,\sin(p^2\,u)}.$$
2. Upon taking partial derivatives with respect to x and p, $p = \cos(x + u)$ yields:
$$0 = (1 + u_x)\,\sin(x + u) \quad \text{and} \quad 1 = -u_p\,\sin(x + u).$$
$$\text{Thus:} \quad u_x = -1 \quad \text{and} \quad u_p = -\frac{1}{\sin(x + u)}.$$
3. Upon taking partial derivatives with respect to x and p, $u = p\,f(x + u)$ yields:
$$u_x = (1 + u_x)\,p\,f'(x + u) \quad \text{and} \quad u_p = f(x + u) + p\,u_p\,f'(x + u).$$
$$\text{Thus:} \quad u_x = \frac{p\,f'(x + u)}{1 - p\,f'(x + u)} \quad \text{and} \quad u_p = \frac{f(x + u)}{1 - p\,f'(x + u)}.$$

8 ExID42. Differentiation within integrals

8.1 Statement: Differentiation within integrals

In each case compute $u_x = \dfrac{\partial u}{\partial x}$ and $u_p = \dfrac{\partial u}{\partial p}$ (as functions of u, x, and p), given that $u = u(x,\,p)$ satisfies:

1. $p = \displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,ds.$

2. $u = \displaystyle\int_0^x \sin\left(p\,u(s^2,\,s) + x\,s\right)\,ds$.

3. $p = \displaystyle\int_x^u \cos\left(p\,\sin(s) + x\,s^2\right)\,ds$.

8.2 Answer: Differentiation within integrals

In each case take partial derivatives of the expressions satisfied by u, and then solve to obtain formulas for u_x and u_p.

1. $1 = \exp\left(p\,\sin(u) + x\,u^2\right)\,u_p + \displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,\sin(s)\,ds$,

so that .... $\boldsymbol{u_p = \dfrac{1 - \displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,\sin(s)\,ds}{\exp\left(p\,\sin(u) + x\,u^2\right)}}$.

$0 = \exp\left(p\,\sin(u) + x\,u^2\right)\,u_x + \displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,s^2\,ds$,

so that .... $\boldsymbol{u_x = -\dfrac{\displaystyle\int_0^u \exp\left(p\,\sin(s) + x\,s^2\right)\,s^2\,ds}{\exp\left(p\,\sin(u) + x\,u^2\right)}}$.

2. Clearly .... $\boldsymbol{u_p = \displaystyle\int_0^x u(s^2,\,s)\,\cos\left(p\,u(s^2,\,s) + x\,s\right)\,ds}$,

and .... $\boldsymbol{u_x = \sin\left(p\,u(x^2,\,x) + x^2\right) + \displaystyle\int_0^x s\,\cos\left(p\,u(s^2,\,s) + x\,s\right)\,ds}$

3. $1 = \cos\left(p\,\sin(u) + x\,u^2\right)\,u_p - \displaystyle\int_x^u \sin(s)\sin\left(p\,\sin(s) + x\,s^2\right)\,ds$,

so that .... $\boldsymbol{u_p = \dfrac{\displaystyle\int_x^u \sin(s)\sin\left(p\,\sin(s) + x\,s^2\right)\,ds}{\cos\left(p\,\sin(u) + x\,u^2\right)}}$.

$0 = \cos\left(p\,\sin(u) + x\,u^2\right)\,u_x - \cos\left(p\,\sin(x) + x^3\right) - \displaystyle\int_x^u \sin\left(p\,\sin(s) + x\,s^2\right)\,s^2\,ds$,

so that .... $\boldsymbol{u_x = \dfrac{\cos\left(p\,\sin(x) + x^3\right) + \displaystyle\int_x^u \sin\left(p\,\sin(s) + x\,s^2\right)\,s^2\,ds}{\cos\left(p\,\sin(u) + x\,u^2\right)}}$.

9 ExID56. Directional derivatives and Taylor

9.1 Statement: Directional derivatives and Taylor

Do the tasks stated in items **1** and **2** below

1. Let Γ be a curve in the plane, $\vec{r} = (x,\, y)$, parameterized by arc-length: $x = X(s)$ and $y = Y(s)$. Assume that $\boldsymbol{\frac{dY}{ds} < 0}$ along the curve, and that the curve is tangent to the unit circle for $s = 0$, at the point $(x,\, y) = (1/\sqrt{2},\, 1/\sqrt{2})$.

Calculate $\boldsymbol{\frac{d\Phi}{ds}}$ *at* $s = 0$*, along the curve* Γ*, for* $\boldsymbol{\Phi = \sin\left(\frac{\pi}{\sqrt{2}}\, x + \pi\, y^2\right)}$.

Correct answer required. "I only missed a sign", or similar, excuses not allowed. **Check** your answer!

2. Let Γ be the straight line in the plane, $\vec{r} = (x,\, y)$, given by $x = 1 + t$ and $y = t$, $-\infty < t < \infty$. Let $\Phi = \Phi(\vec{r})$ be some smooth scalar function. Define $f = f(t)$ by $f = \Phi$ along Γ.

Write the first three terms of the Taylor expansion for f *at* $t = 0$*, in terms of the partial derivatives of* Φ *at* $\vec{r}_0 = (1,\, 0)$*. In particular, compute* $\dot{f}(0)$ *and* $\ddot{f}(0)$ *for* $\Phi = x^2\, e^y$.

9.2 Answer: Directional derivatives and Taylor

We have

1. At $s = 0$, $\boldsymbol{\frac{d\vec{r}}{ds}}$ along Γ must be a unit tangent vector to the unit circle at $\vec{r} = (1/\sqrt{2},\, 1/\sqrt{2})$. That is

$$\text{either (i)}\quad \frac{d\vec{r}}{ds} = (1/\sqrt{2},\, -1/\sqrt{2}) = \hat{t}_1 \quad \text{or (ii)}\quad \frac{d\vec{r}}{ds} = (-1/\sqrt{2},\, 1/\sqrt{2}) = \hat{t}_2.$$

However, since $\boldsymbol{\frac{dY}{ds} < 0}$, it must be (i). Thus, at $s = 0$,

$$\frac{d\Phi}{ds} = \frac{d\vec{r}}{ds}\cdot\nabla\,\Phi = \frac{1}{\sqrt{2}}\,\Phi_x\left(\frac{1}{\sqrt{2}},\,\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\,\Phi_y\left(\frac{1}{\sqrt{2}},\,\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{2}\,\cos(\pi) = \frac{\pi}{2}. \tag{9.1}$$

2. The tangent vector to Γ is $\dfrac{d\vec{r}}{dt} = (1,\, 1) = \vec{t}$. Thus

$$\begin{aligned} f &= \Phi_0 + t\,\left((\vec{t}\cdot\nabla)\;\Phi\right)_0 + \frac{1}{2}\,t^2\,\left((\vec{t}\cdot\nabla)^2\;\Phi\right)_0 + \dots \\ &= \Phi_0 + t\,(\Phi_x + \Phi_y)_0 + \frac{1}{2}\,t^2\,(\Phi_{xx} + 2\,\Phi_{xy} + \Phi_{yy})_0 + \dots \end{aligned} \tag{9.2}$$

where the subscript zero indicates evaluation at $\vec{r}_0$. In particular, if $\Phi = x^2\, e^y$,

$$f = 1 + 3\,t + \frac{7}{2}\,t^2 + \dots \tag{9.3}$$

so that $\dot{f}(0) = 3$ and $\ddot{f}(0) = 7$.

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10 ExID61. Direct Taylor expansions

10.1 Statement: Direct Taylor expansions

For the examples below, calculate the Taylor expansion up to the order indicated (e.g.: $\cos(x) = 1 - \frac{1}{2}\,x^2 + O(x^4)$). Do not use a calculator to evaluate constants that appear in the expansions — e.g., $\sqrt{2}/\pi$ or $\cos(3)$. On the other hand, do simplify when possible — e.g., $\tan(\pi/4) = 1$ or $2/\sqrt{2} = \sqrt{2}$.

1. Expand, up to $O(x^4)$, $f(x) = \sin(x)\,\cos(\sqrt{x})$.

2. Expand, up to $O(x^5)$, $f(x) = \sin(1+x)$.

3. Expand, up to $O(x^5)$, $f(x) = \sin(1+x+x^3)$.

4. Let $G = G(x,\,y)$ be some smooth[7] function of two variables. For $z \geq 0$, expand up to $O(z^3)$, $f(z) = G(z,\,z^{1.5})$. Express the expansion coefficients in terms of partial derivatives of G.

10.2 Answer: Direct Taylor expansions

We have

1. $f(x) = \left(x - \dfrac{1}{6}\,x^3 + O(x^5)\right)\,\left(1 - \dfrac{1}{2}\,x + \dfrac{1}{24}\,x^2 + O(x^3)\right) = x - \dfrac{1}{2}\,x^2 - \dfrac{1}{8}\,x^3 + O(x^4)$.

2. $f(x) = \sin(1) + \cos(1)\,x - \dfrac{1}{2}\,\sin(1)\,x^2 - \dfrac{1}{6}\,\cos(1)\,x^3 + \dfrac{1}{24}\,\sin(1)\,x^4 + O(x^5)$.

3. Using the answer to part **2**, we get

$$\begin{aligned} f(x) &= \sin(1) + \cos(1)\,(x+x^3) - \tfrac{1}{2}\,\sin(1)\,(x^2 + 2\,x^4 + O(x^6)) - \\ &\quad \tfrac{1}{6}\,\cos(1)\,(x^3 + O(x^5)) + \tfrac{1}{24}\,\sin(1)\,(x^4 + O(x^6)) + O(x^5) \\ &= \sin(1) + \cos(1)\,x - \tfrac{1}{2}\,\sin(1)\,x^2 + \tfrac{5}{6}\,\cos(1)\,x^3 - \tfrac{23}{24}\,\sin(1)\,x^4 + O(x^5). \end{aligned}$$

4. We have $G(x,\,y) = G^0 + G^0_x\,x + G^0_y\,y + \frac{1}{2}\,G^0_{x\,x}\,x^2 + G^0_{x\,y}\,x\,y + \frac{1}{2}\,G^0_{y\,y}\,y^2 + O\left((x^2+y^2)^{1.5}\right)$,
where: The superscript 0 denotes evaluation at $(0,\,0)$ — e.g., $G^0 = G(0,\,0)$, etc.
The subscripts denote partial derivatives — e.g., $G_x = \frac{\partial G}{\partial x}$, etc.

Thus $f(z) = G^0 + G^0_x\,z + G^0_y\,z^{1.5} + \dfrac{1}{2}\,G^0_{x\,x}\,z^2 + G^0_{x\,y}\,z^{2.5} + O(z^3)$.

11 ExID71. Change of variables for an ode

11.1 Statement: Change of variables for an ode

Consider the second order, nonlinear, ode

$$x^2\,\cos w\,\frac{d^2w}{dx^2} - x^2\,\sin w\,\left(\frac{dw}{dx}\right)^2 + x\,\cos w\,\frac{dw}{dx} + \sin w = 0 \tag{11.1}$$

for $w = w(x)$, where $x > 0$. Rewrite it in terms of $u = u(y)$, where $u = \sin w$ and $y = \ln x$.

11.2 Answer: Change of variables for an ode

From the relationship between y and x, and the chain rule, it follows that $\boldsymbol{\dfrac{d}{dy} = x\,\dfrac{d}{dx}}$.

Hence

$$\frac{du}{dy} = x\,\cos w\,\frac{dw}{dx}, \tag{11.2}$$

$$\frac{d^2u}{dy^2} = x\,\cos w\,\frac{dw}{dx} - x^2\,\sin w\,\left(\frac{dw}{dx}\right)^2 + x^2\,\cos w\,\frac{d^2w}{dx^2}, \tag{11.3}$$

[7] The partial derivatives of f, to any order, exist and are continuous

where the second equation follows from expanding $\dfrac{d^2u}{dy^2} = x\,\dfrac{d}{dx}\left(x\,\cos w\,\dfrac{dw}{dx}\right)$. Equation (11.2) could be used to write $\frac{dw}{dx}$ in terms of $\frac{du}{dy}$. Then (11.3) would yield $\frac{d^2w}{dx^2}$ in terms of $\frac{du}{dy}$ and $\frac{d^2u}{dy^2}$. Finally, these expressions, substituted into (11.1), would produce the final answer. However, here it is easy to see directly from (11.3) that the transformed equation is

$$\frac{d^2u}{dy^2} + u = 0. \tag{11.4}$$

12 ExID77. Change of variables for a pde

12.1 Statement: Change of variables for a pde

Let $u = u(x,\,t)$ be a solution of the heat

$$u_t = u_{xx}. \tag{12.1}$$

What equation does $\phi = -\frac{1}{u}\,u_x$ satisfy?

Hint. Calculate ϕ_t and use the equation for u. Calculate ϕ_x and write it in terms of u, u_{xx}, and ϕ^2. Then compute ϕ_{xx}. You should now be able to write ϕ_t in terms of ϕ, ϕ_x, and ϕ_{xx}.

12.2 Answer: Change of variables for a pde

We have

$\phi_t = -\frac{1}{u}\,u_{xt} + \frac{1}{u^2}\,u_x\,u_t = -\frac{1}{u}\,u_{xxx} + \frac{1}{u^2}\,u_x\,u_{xx}$, where we have used (12.1).

$\phi_x = -\frac{1}{u}\,u_{xx} + \phi^2$.

$\phi_{xx} = -\frac{1}{u}\,u_{xxx} + \frac{1}{u^2}\,u_x\,u_{xx} + (\phi^2)_x$.

Thus ϕ satisfies

$$\phi_t + (\phi^2)_x = \phi_{xx}. \tag{12.2}$$

THE END.