Laplace Transform Facts

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Defined for f integrable with at worse exponential growth for $t \to \infty$.

1) Analytic for $\mathrm{Re}(s) > C_a$ = constant

2) $\mathcal{L}(f') = -f(0) + sF$ (integrate by parts)

3) $\mathcal{L}(f'') = -f'(0) - sf(0) + s^2F$, etc.

4) Inverse $f(t) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma F(s)e^{st}\,ds$

Γ = path $s = \mu + ik$, μ = const. $> C_a$ and $-\infty < k < \infty$, with direction k growing.

Example 1

$u_{tt} - u_{xx} = 0 \quad 0 < x < 1$

with $u = 0$ at $x = 0, 1$ and

$u = f(x)$, $u_t = g(x)$ for $t = 0$.

Then $-g - sf + s^2U - U'' = 0$

with $U = U(s, x)$, $U(s,0) = U(s,1) = 0$,

and $' = \dfrac{\partial}{\partial x}$

That is

$$-U'' + s^2 U = g + sf$$

The Green's function $G(x,y,s)$ for this problem, defined by

$$-G_{xx} + s^2 G = \delta(x-y)$$

+ boundary Cond. ($G=0$ at $x=0,1$) is

$$G = \frac{-1}{s\sinh(s)} \begin{cases} \sinh(sx)\sinh(s(y-1)) & x<y \\ \sinh(sy)\sinh(s(x-1)) & x>y \end{cases}$$

Then $\displaystyle U = \int_0^1 G(x,y)\{g(y) + sf(y)\}\,dy$

<u>Note</u>: Singularities of G at $s = in\pi$, $n \neq 0$

These are simple poles (explain why $n=0$ is not singular)

<u>Residue at $s = in\pi$</u>

$$G_n = \frac{-1}{in\pi\cosh(in\pi)} \begin{cases} \sinh(s_n x)\sinh(s_n(y-1)) & x<y \\ \sinh(s_n y)\sinh(s_n(x-1)) & x>y \end{cases}$$

$$s_n = in\pi$$

<u>Recall</u> $\sinh(iz) = +i\sin(z)$, $\cosh(iz) = \cos z$

LT 2a

$G = \alpha \sinh(sx)$ for $x < y$

$\beta \sinh(s(x-1))$ for $x > y$

Continuity at $x = y$ $\quad \alpha \sinh(sy) = \beta \sinh(s(y-1))$

$$\therefore G = \begin{cases} \mu \sinh(sx) \sinh(s(y-1)) & x<y \\ \mu \sinh(sy) \sinh(s(x-1)) & x>y \end{cases}$$

δ at $x = 1$ require

$$+G_x^- \mp G_x^+ = 1 \quad \text{at } x = y$$

i.e.

$$s\mu\left[+\cosh(sy)\sinh(s(y-1)) \mp \sinh(sy)\cosh(s(y-1))\right] = 1$$

Wronskian constant! Evaluate at $y = 1$

$$\boxed{-\mu s \sinh(s) = 1}$$

Explain Wronskian and Green's functions?

LT2b

Green's function $s=0$

take limit!!

$$G = \begin{cases} -x(y-1) & x<y \\ -y(x-1) & x>y \end{cases}$$

Thus

$$G_n = \frac{1}{i n\pi (-1)^n} \begin{cases} \sin(n\pi x) \sin(n\pi(y-1)) & x<y \\ \sin(n\pi y) \sin(n\pi(x-1)) & x>y \end{cases}$$

$\therefore$ Residue for U $\Big] = \frac{1}{i n\pi} \sin(n\pi x) \sin(n\pi y)$

$$U_n = \frac{1}{i n\pi} \underbrace{\int_0^1 \sin(n\pi y)\, g(y)\, dy}_{g_n} \sin(n\pi x)$$

$\boxed{\text{case } f \equiv 0}$

$$U_n = + \underbrace{\int_0^1 \sin(n\pi y) f(y) dy}_{f_n} \sin(n\pi x)$$

$\boxed{\text{Case } g \equiv 0}$

Evaluate Inverse L.T. using residue theorem [see LT4!]

$$\cancel{u = 2\pi i \sum U_n = \sum_{n\neq 0} \frac{2}{n} g_n \sin(n\pi x) e^{in\pi t}}$$

$$\cancel{+ \sum_{n \neq 0} (2\pi i) f_n \sin(n\pi x) e^{in\pi t}}$$

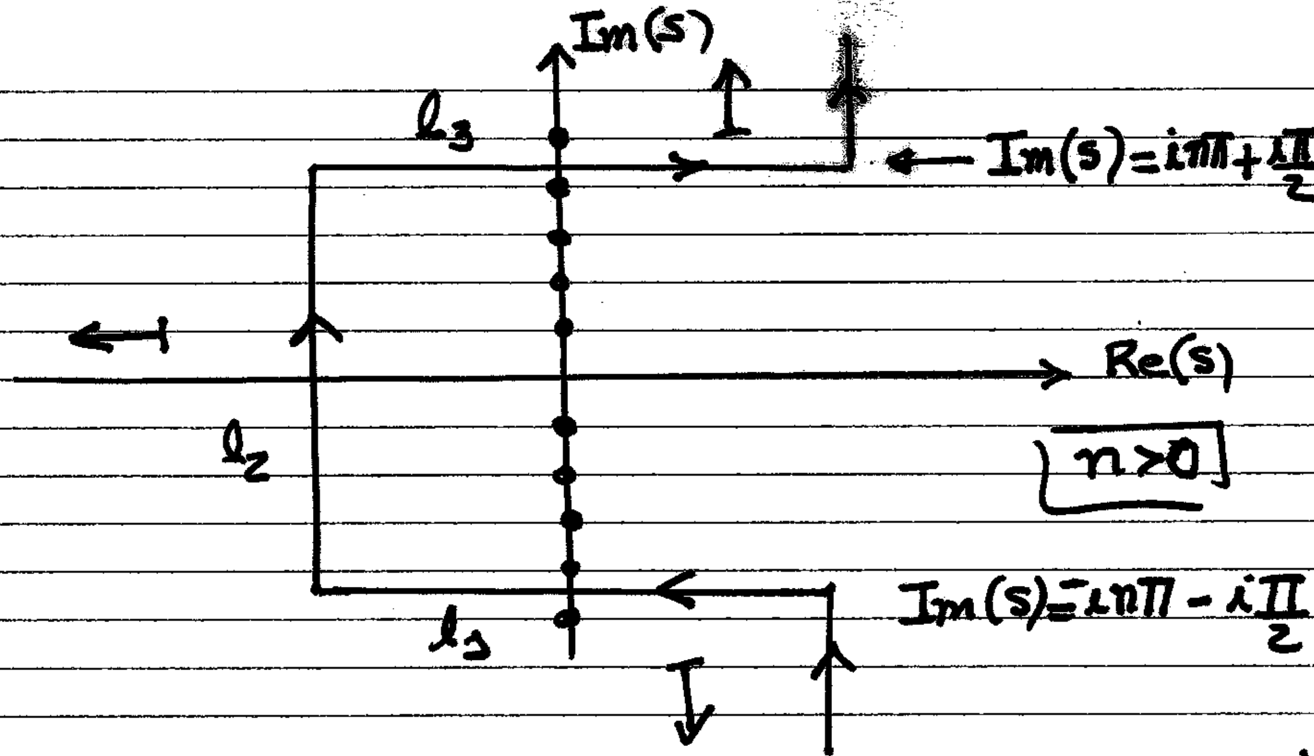
Note: $g_{-n} = -g_n$ & $f_{-n} = f_n$

$$u = \sum U_n = \sum_{n\neq 0} \frac{1}{in\pi} g_n \sin(n\pi x) e^{in\pi t}$$

$$+ \sum_{n\neq 0} f_n \sin(n\pi x) e^{in\pi t}$$

$$= \sum_{n=1}^{\infty} 2 g_n \sin(n\pi x) \frac{\sin(n\pi t)}{n\pi} + 2 f_n \sin(n\pi x) \cos(n\pi t)$$

Fourier Series Solution.



The integrand is $I = \dfrac{-1}{s\sinh(s)}\,\underset{x<y}{\sinh(sx)\sinh(s(y-1))}\,e^{st}$ etc

Over the segments ℓ_1 and ℓ_3 $\sinh(s)$ is independent of n (periodicity) and the numerator stays bounded. Thus, because of the $\frac{1}{s}$ factor these integrals vanish as $n \to \infty$

Consider now the integral over ℓ_2 (after the $n \to \infty$ limit) as $\mathrm{Re}(s) \to -\infty$

Then $|\sinh(sx)| \le C\, e^{-\mathrm{Re}(s)x}$

$$|\sinh(s(y-1))| \le C\, e^{-\mathrm{Re}(s)(1-y)}$$

$$|\sinh(s)| \sim \frac{1}{2} e^{-\mathrm{Re}(s)}$$

Thus $|I| \lesssim \dfrac{C}{|s|} \underbrace{e^{-\mathrm{Re}(s)[1-y+x-1]}}_{e^{\mathrm{Re}(s)(y-x)}} e^{+\mathrm{Re}(s)t}$

but $y > x$ $\therefore$ this vanishes exponentially

$\Rightarrow$ only residues remain!

$$\boxed{\text{Very important to check integrals vanish, so that only residues remain}}\ !$$

__Example 2__ $u_t + u_x = 0 \quad 0 < x < 1$

$u = 0$ for $x = 0$, $u = f(x)$ for $t = 0$

Then
$$-f + sU + U' = 0, \quad U = 0 \text{ for } x = 0$$

i.e. $U' + sU = f$ or $U = e^{-sx} \underline{\int_0^x e^{sy} f(y)\,dy}$

Note: U has no singularities!

$\therefore$ if we could move the integral over Γ in the inverse to $\mathrm{Re}(s) = -\infty$, we would get $u = 0$; nonsense!!

But $e^{-s(x-y)}$ blows up exponentially as $\mathrm{Re}(s) \to -\infty$; cannot discard the integral.

This example is interesting because it <u>does not have normal modes</u>

[generally: singularities of the L.T. are associated with <u>spectrum</u>] †

Equation for eigenfunctions; $u = \varphi(x)e^{\lambda t}$

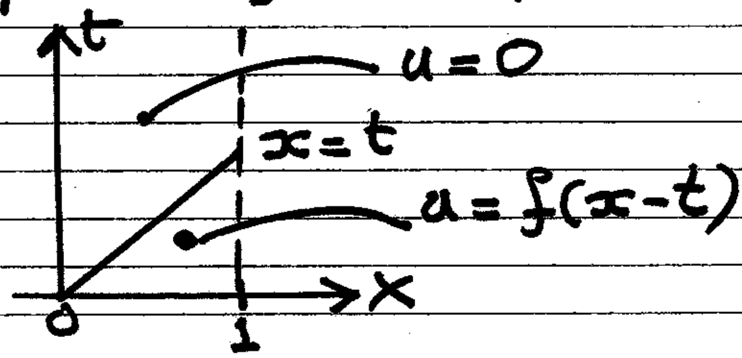
$$\varphi' = -\lambda\varphi \quad , \quad \varphi(0) = 0$$

$\therefore\ \varphi = c e^{-\lambda x}$ but $\Rightarrow c = 0$

<u>No</u> eigenmodes

{ This is an example where an ∞ operator has no spectrum ... this is why "normal" is so important for ∞ operators!

<u>What is going on?</u>



Solution vanishes after finite time!
But eigenmodes can only vanish exponentially!

Example 3 Mass at the end of a string

Of course example 1 was kind of too simple! We knew the answer in advance. Now:

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < 1$$
$$u = 0 \qquad x = 0$$
$$m u_{tt} = -ku - T u_x \qquad x = 1$$

and again $u = f$, $u_t = g$ for $t = 0$

Then
$$-U'' + s^2 U = g + sf \qquad 0 < x < 1$$
$$U = 0 \quad \text{for } x = 1$$

and, at $x = 1$:
$$s^2 U - u_t^0 - s u^0 = -\frac{k}{m} U - \frac{T}{m} U'$$

We split the problem into ~~[illegible]~~

~~[illegible]~~

Solution with $g + sf = 0$ (a)

" with $u_t^0 + s u^0 = 0$ (b)

The solution to (b) is similar to example 1, but the B.C. at $x = 1$ is $\left(s^2 + \frac{k}{m}\right) U + \frac{T}{m} U' = 0$ (Robin)

Green's function is ~~[illegible]~~ harder ~~[illegible]~~ to write, but same idea works.

For (a) we try

$$U = \mu \sinh(s x), \quad \text{some const. } \mu.$$

Substituting into the B.C.

$$\mu\left(s^2 + \frac{k}{m}\right)\sinh(s) + \mu \frac{I}{m} s \cosh(s) = u_t^0 + s u^0$$

This determines μ. The inverse transform is then done with the same method

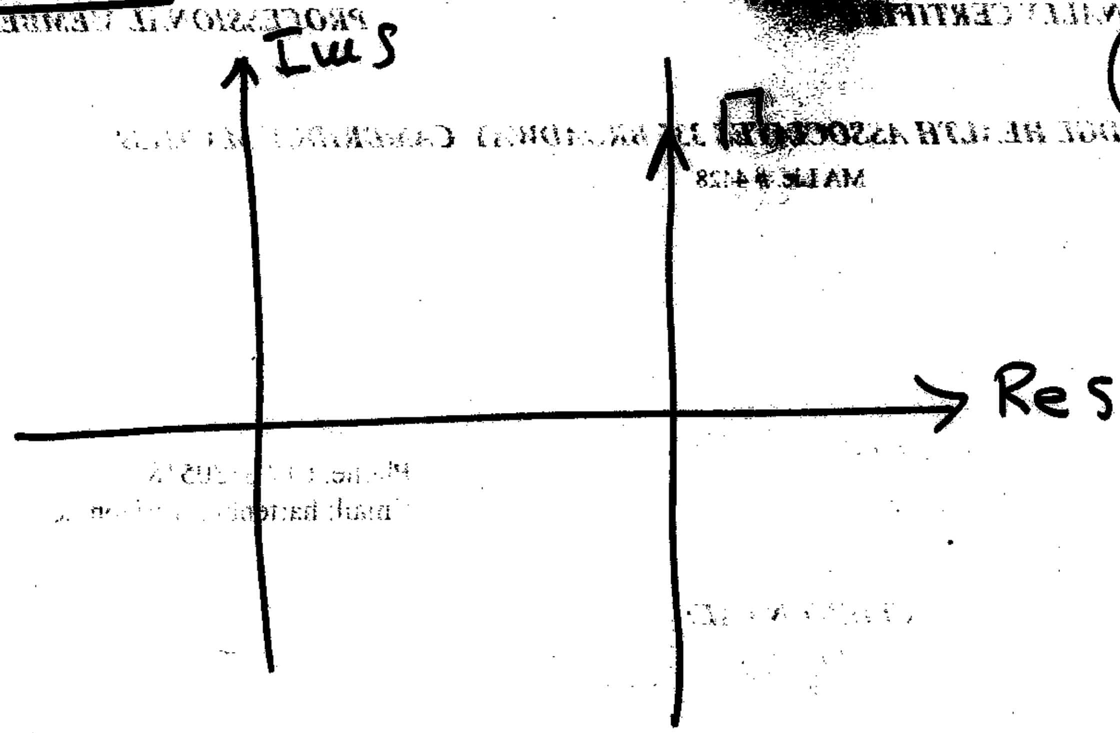
{ Note that singularities in μ now correspond to resonances.

<u>Note</u>: Continuum spectrum, etc. happens when U has singularities other than poles [essential or branch points].

Using the L.T. is not restricted to self-adjoint or normal problems (nor constant coeff.) "Clever" manipulations with it allow you to extract information about the solution otherwise you would not have.

Example (Combustion, [illegible]drops)

LT9



Suppose you can move Γ to $Re(s) < 0$

(but not $Re(s) = -\infty$!)

Then stability: no singularities on $Re(s) > 0$

unstability: singularities on $Re(s) > 0$

Such singularities can be reached numerically in a stable way,

← Far better than discretizing and then trying to look at a finite approx. to problem ← can be very unstable