Fourier Transforms & QM uncertainty principle (1)

a) In QM, if $\psi = \psi(\vec{x}, t)$ is the wave function (solution of Schrödinger's equation), $\rho = |\psi|^2$ is the pdf for the particle position.

b) de Broglie's equation states that for a plane wave $\psi = e^{ikx}$, the momentum is $p = \hbar k$. We can re-state this as: for $\psi = \frac{e^{ikx}}{\alpha}$, the momentum wave† function is $\varphi = \delta(p - \hbar k)$

c) In general $\psi = \frac{1}{\sqrt{2\pi}} \int \hat{\psi}(k)\, e^{ikx}\, dk$

$\Rightarrow$

$$\varphi(p) = \frac{\alpha}{\sqrt{2\pi}} \int \delta(p - \hbar k)\, \hat{\psi}(k)\, dk$$

$$= \frac{\alpha}{\hbar\sqrt{2\pi}}\, \hat{\psi}\left(\frac{p}{\hbar}\right)$$

---

† α is a normalization constant that we need to introduce to make sense of probabilities

Using Plancherel's theorem

$$\int |\varphi(p)|^2 dp = \frac{\alpha^2}{\hbar^2 2\pi} \int |\hat{\psi}(\tfrac{p}{\hbar})|^2 dp$$

$$= \frac{\alpha^2}{\hbar 2\pi} \int |\hat{\psi}(k)|^2 dk$$

$$= \frac{\alpha^2}{\hbar 2\pi} \underbrace{\int |\psi(x)|^2 dx}_{1}$$

$\therefore \; \boxed{\alpha = \sqrt{2\pi\hbar} = \sqrt{h}}$

So momentum wave-function $\Big\} \; \boxed{\dfrac{1}{\sqrt{h}}\hat{\psi}\left(\dfrac{p}{\hbar}\right)}$

Note in 3-D we get $\boxed{\varphi = \dfrac{1}{h^{3/2}}\hat{\psi}\left(\dfrac{\vec{p}}{\hbar}\right)}$

note h, not $\hbar$

Dimensional Analysis

$[\psi] = 1/\ell^{3/2} \Rightarrow [\hat{\psi}] = \ell^{3/2}$

Thus $[\varphi] = \left(\dfrac{\ell}{[h]}\right)^{3/2} = \left(\dfrac{t}{m\ell}\right)^{3/2} = 1/[p]^{3/2}$ ✓

since $[h] = m\ell^2/t$

<u>Note now</u> $\quad F(x) = f\left(\frac{x}{\epsilon}\right)$

$$\Rightarrow \quad \hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(\frac{x}{\epsilon}\right) e^{-ikx}\, dx$$

$$= \frac{\epsilon}{\sqrt{2\pi}} \hat{f}(\epsilon k)$$

Thus, as the certainty in position grows ($\epsilon \downarrow 0$), the uncertainty in momentum grows. Heisenberg's principle $\quad \Delta x \, \Delta p \geq h/\cancel{4}\pi = \frac{1}{2}\hbar$

follows from an inequality relating the covariance of a function and its F.T.