Summary. This final problem set of the term contains questions regarding viscous fluid mechanics and some elementary questions on solid mechanics. Comments and corrections should be e-mailed to: thomsons@mit.edu.

You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

FLUIDS

P1.1) Non-dimensionalisation of the Navier-Stokes equations

Consider viscous flow past an object with a characteristic length scale $L$, with far-field velocity $U$, where the fluid flow is governed by the incompressible Navier-Stokes equations.

(i) Deduce that one possible scaling for the pressure is $\rho U^2$. Hence non-dimensionalise the incompressible Navier-Stokes equations and the vorticity transport equation (Problem Set 4, equation (4)). Show that there is a single dimensionless parameter, the Reynolds number $Re = \rho uL/\mu = uL/\nu$.

(ii) Comment on the physical significance of the Reynolds number with respect to the ratio of (a) inertial to viscous forces and (b) the timescales of convection and diffusion of vorticity.

(iii) Deduce that there is another possible pressure scaling $\mu U/L$, and non-dimensionalise the equations from Part (i). Which form of the dimensionless equations is more relevant when Re is (a) large (high Reynolds number flow) and (b) small (Stokes flow).

(iv) Compare the sizes of the Reynolds number for (a) air flow past an aeroplane wing in flight and (b) swimming bacteria in water.

P1.2) Diffusion of vorticity in plane parallel shear flow

Consider a viscous fluid confined below by a rigid plate at $y = 0$. At $t = 0$, the plate is suddenly set into motion with velocity $\mathbf{v}_B = (U, 0, 0)$. We assume that the flow is being driven by the motion of the boundary, and not by any externally applied
pressure gradient. As shown in lecture, the velocity component
\[ u = U \text{erfc} \left( \frac{y}{2\sqrt{\nu t}} \right), \]
the function \text{erfc}(x) being the complementary error function.

(i) Find the distance \( \delta \) over which the velocity has decayed to 1% of the boundary velocity \( U \) after time \( t \). Plot the solutions \( u \) (using e.g. MATLAB, Mathematica) for different times \( t \) in the case \( U = \nu = 1 \), and comment on how the flows evolves as \( t \to \infty \).

(ii) Finally, show that the vorticity \( \omega = (0, 0, \omega(y, t)) \) and determine the function \( \omega \). Comment on the diffusion of vorticity throughout the fluid for \( t \in (0, \infty) \), and argue that vorticity diffuses a distance of order \( L \) in a time \( \tau \sim L^2/\nu \) (the time scale of viscous diffusion). Indicate on your plots from Part (i) the layer in which vorticity is significant.

**P1.3)** Consider an analogous problem to that describe in P1.2), only now the plate at \( y = 0 \) oscillates with a prescribed velocity \( \mathbf{v}_B = (U \cos(\Omega t), 0, 0) \). You may assume again that the fluid is driven by the movement of the plate and that the fluid velocity \( \mathbf{v} = (u(y, t), 0, 0) \).

(i) Show that the Navier-Stokes equations may be reduced to a diffusion equation for the velocity \( u \) in the absence of an applied pressure gradient. Write down the relevant initial and boundary conditions describing the flow.

(ii) By seeking a solution to the problem you constructed in Part (i) of the form \( u = \text{Re}(U f(y) \exp(i\Omega t)) \), show that
\[ u = U \exp(-\gamma y) \cos(\gamma y - \Omega t), \]
where \( \gamma = \sqrt{\Omega/2\nu} \).

(iii) Sketch or plot the solution \( u \) as a function of \( y \) for different times \( t \) (starting at \( t = 0 \)), describing your observations. Then compute the vorticity \( \omega \) and show that it is exponentially small except in a (Stokes) layer of size \( \sqrt{\nu/\Omega} \). What happens to the thickness of this layer as the frequency \( \Omega \) increases? Does this correspond to your physical intuition?

**Solids**

**P2.1)** The displacement corresponding to a rigid-body motion is given by
\[ \mathbf{u} = \mathbf{c}(t) + (\mathbf{Q}(t) - \mathbf{I}) \mathbf{X}, \]
where the orthogonal matrix \( \mathbf{Q} \) and vector \( \mathbf{c} \) are spatially uniform. The matrix \( \mathbf{I} \) is the identity matrix. Show that the deformation gradient \( \mathbf{F} \) is orthogonal for this

\[ \text{Here we are assuming that, after long time, the solution is periodic. In principle, the problem constructed in Part (i) is soluble using a Laplace transform (in case you are tempted to try it).} \]
displacement field, and hence that the strain tensor
\[ E = \frac{1}{2} (F^T F - I) \equiv 0. \]
On the other hand, show that the *linearised* strain tensor
\[ e = \frac{1}{2} (\nabla u + \nabla u^T) \]
in general does not vanish identically.

**P2.2)** The Cauchy momentum equation in Lagrangian coordinates (neglecting body forces) is given by
\[ \rho_0 \frac{\partial^2 u}{\partial t^2} = \nabla \cdot T, \]
where the first Piola-Kirchhoff stress tensor \( T = J \sigma (F^T)^{-1} \) and \( \sigma \) is the Cauchy stress tensor. As shown in lecture, the second Piola-Kirchhoff stress tensor \( S = F^{-1} T \).
Consider the unidirectional displacement \( u(X, t) \) in the \( X \)-direction. Assuming the constitutive relation
\[ S = (\lambda + 2\mu) E_{xx}, \]
show that the momentum equation becomes
\[ \frac{\partial^2 u}{\partial t^2} = c_p^2 G'(\frac{\partial u}{\partial X}) \frac{\partial^2 u}{\partial X^2}, \quad (1) \]
where \( G(u_X) = (1 + u_X)(1 + u_X/2)u_X \) and \( c_p^2 = (\lambda + 2\mu)/\rho_0. \)
Finally, show that, upon defining \( p = \partial u/\partial t \) and \( q = \partial u/\partial X \), the wave equation \( (1) \) may be recast as the following 2 × 2 system of p.d.e.
\[ \frac{\partial q}{\partial t} - \frac{\partial p}{\partial X} = 0, \quad (2a) \]
\[ \frac{\partial p}{\partial t} - c_p^2 G'(q) \frac{\partial q}{\partial X} = 0. \quad (2b) \]
Now, write the system \( (2) \) as a matrix-vector system of the form
\[ A u_t + B u_X = 0, \quad (3) \]
where \( u = (p, q)^T \). The characteristics \( \dot{X} \) of \( (1) \) may be determined by computing \( \det(B - \dot{X} A) = 0. \) Hence show that the characteristics of \( (1) \) are given by
\[ \dot{X} = \pm c_p^2 \sqrt{1 + \frac{3}{2} \frac{\partial u}{\partial X} \left( \frac{\partial u}{\partial X} + 2 \right)}. \]
Comment on the shock-forming potential of equation \( (1) \) given the form of the characteristics determined above.

**THE END**