You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

**Dimensional analysis**

P1.1) **Drag on a sphere in Stokes flow**

Through careful observation and experiment, it is supposed that the drag \( f \) on a sphere depends on its diameter \( D \), the relative velocity of the sphere to the fluid \( U \), and the fluid density and dynamic viscosity \( \rho \) and \( \mu \), respectively. In other words

\[
 f = f(D, U, \rho, \mu) \quad \text{or} \quad g(f, D, U, \rho, \mu) = 0.
\]

What are the independent dimensions involved in the problem? Hence, by appealing to the Buckingham-\( \Pi \) theorem, deduce the number of independent dimensionless groups in the problem.

Determine now what these dimensionless groups are, and hence provided an expression for the drag on a sphere.

P1.2) **Surface tension**

Consider a fluid of density \( \rho \), kinematic viscosity \( \nu \), and surface tension \( \sigma \). Suppose some flow is endowed with a characteristic length scale \( L \) and velocity scale \( U \), and is subject to gravitational acceleration \( g \).

By considering the dimensions of the various physical quantities at play, how many dimensionless groups characterise the fluid? Find these dimensionless groups and offer a physical interpretation for each i.e. what physical effects do the parameters compare?

P1.3) **Aquatic locomotion**

Read the paper *Scaling macroscopic aquatic locomotion*, by Gazzola, Argentina, and Mahadevan, Nature Physics (2014). In particular,
(a) Write down the physical variables thought to be important to swimming speed $U$, and their corresponding units and dimensions.

(b) Verify that the parameters Re and Sw are, indeed, dimensionless.

(c) Summarise how the scaling parameter $\alpha$ is determined for both laminar and turbulent flows.

(d) Comment on the efficacy of these scaling relationships in describing experimental data (Figures 2 & 3), and why these scaling laws are thought to work so well.

**Perturbation methods**

P2.1) **Algebraic equations**

By rescaling $x$ or otherwise, find the asymptotic expansions as $\varepsilon \to 0$ of the roots of

(a) (Bender and Orzag, Problem 7.5(a))

$$\varepsilon x^3 + x^2 - 2x + 1 = 0$$

*to three terms;*

(b) (Thomson, Durey, & Rosales, 2020)

$$\varepsilon x^3 + (\varepsilon + 1)x^2 + (a\varepsilon^2 + 1)x + \varepsilon(a - b) = 0$$

*to one term.*

P2.2) Is the following problem (Bender and Orzag, Problem 7.4) regular or singular?

$$x^3 - x^2 + \varepsilon = 0$$

Justify your choice and determine the **first three terms** in the asymptotic expansions of its roots as $\varepsilon \to 0$.

P2.3) **Differential equations**

(a) Consider the following differential equation

$$y''(x) + 2y'(x) + \varepsilon y(x) = 0, \quad \text{subject to} \quad y(0) = 0, \ y(1) = 2.$$  

(1)

(i) Solve this problem exactly.

(ii) Using a perturbation expansion of the form

$$y = \sum_{j=0}^{\infty} \varepsilon^j y_j(x),$$

find the **first two terms** in the asymptotic expansion of (1) as $\varepsilon \to 0$.

(iii) Should this be regarded as a singular or regular perturbation problem as $\varepsilon \to 0$?

(b) Consider the following differential equation

$$\varepsilon y'' + p(x)y' + y = 0 \quad \text{subject to} \quad y(0) = 0, \ y(1) = 1.$$  

(2)
Do you think that this is a regular or singular perturbation problem?

First suppose that \( p(x) = -\sqrt{x} \).

(i) By rescaling \( x = \varepsilon^\beta X \) in (2), and considering what happens as \( X \to \infty \) to the leading-order inner solution (when \( \varepsilon = 0 \)), argue that there cannot be a boundary layer at \( x = 0 \).

(ii) Show that the outer solution when \( x = O(1) \) is \( y(x) \equiv 0 \).

(iii) By rescaling \( x = 1 - \varepsilon^\gamma X \), find the \( \gamma \) that transforms equation (2) to a regular problem. Show that calculating the first two terms of the inner solution \( Y(X) \) leads to
\[
Y(X) = \exp(-X) + \varepsilon \frac{X}{4}(X + 6) \exp(-X) + O(\varepsilon^2).
\]

(iv) Comment on the matching of your inner solution \( Y \) to the outer solution \( y \).

Finally, what happens when \( p(x) = 0 \)? That is, where (if anywhere) is the boundary layer. Hint: solve (2) exactly in this case.