Summary. This problem set reviews some of the basic mathematical tools that will be useful as we proceed through the course. Some, if not all, of these concepts will hopefully be familiar to you from previous courses but if not, I encourage you to revisit old lecture notes etc. to make sure you are comfortable with what is being assessed here. Comments and corrections should be e-mailed to: thomsons@mit.edu.

You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

P1.1) One- and two-variable implicit differentiation

In each of the following examples, find $y' = \frac{dy}{dx}$ as a function of $y$ and $x$ given that $y(x)$ satisfies:

(a) $x^3 + 3xy + y^2 = 0$
(b) $y = \tan \left( x + \frac{1}{2} y^2 \right)$
(c) $\ln(1 + y) = e^x$

(d) $y^2 = a^x$
(e) $y = \frac{1}{f(x + y)}$
(f) $y = f(1 + xy)g(x + y)$.

Now do the same in the following examples, only this time find both $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$ as functions of $u, x$ and $y$, given that $u(x, y)$ satisfies:

(g) $x^3y + 3yu + yu^3 = 0$
(h) $\cos(y^2u) = ye^{-x^2}$
(i) $\ln(1 + y) = ue^{xu}$.

P1.2) Differentiation within integrals (a.k.a. Leibnitz’ rule)

For each of the following evaluate $u_x$ and $u_y$ as functions of $u, x$, and $y$ given that $u(x, y)$ satisfies:

(a) $u = \int_{a(x)}^{b(x)} f(x, y, s) \, ds$
(b) $y = \int_{x}^{u} \exp \left( y\sin(s) + xs^2 \right) \, ds$
(c) $u = \int_{0}^{x} \sin(yu(s^2, s) + xs) \, ds$
P1.3) Single- and multi-variate Taylor expansions

Find the Taylor expansion about $x = 0$ up to order $O(x^5)$ for each of the following functions:

(a) $f(x) = e^x \cos(x)$
(b) $f(x) = \sin(1 + x)$
(c) $f(x) = \sin(1 + x + x^3)$

Now state the general form for the Taylor series of a function $f$ of two variables $x$ and $y$ (i.e. $f = f(x,y)$) about the point $(x, y) = (x_0, y_0)$ up to and including quadratic terms. Use this expression to evaluate the Taylor expansion of $f(x, y) = y \exp xy$ about the point $(x, y) = (2, 3)$.

P1.4) Complex numbers

(a) Express the following complex numbers in polar exponential form i.e. in the form $z = \rho \exp(i\theta)$.
(i) $1 + i$
(ii) $\frac{1}{2} \left(1 + \sqrt{3}i\right)$

(b) Express the following complex numbers in the form $x + iy$, where $x, y \in \mathbb{R}$.
(i) $(1 + i)^3$
(ii) $\exp \left(2 + \frac{i\pi}{2}\right)$
(iii) $\cos \left(\frac{i\pi}{4} + c\right)$, where $c \in \mathbb{R}$.

*Hint:* For (b) (iii), you may use the fact that $\cos(z) = (\exp(iz) + \exp(-iz))/2$ and $\exp(z) = \exp(x) \exp(iy)$.

(c) Find all the (complex) roots of the following equations.
(i) $z^3 = 4$
(ii) $z^4 = -1$
(iii) $z^4 + 2z^2 + 2 = 0$

(continued over)

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1This notation means, for example, $\cos x = 1 - x^2/2 + O(x^4)$
P1.5) Canonical forms on linear p.d.e.

A partial differential equation which may already be familiar to you is the linear wave equation, namely
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \]  
(1)

where \( u = u(x, t) \) and \( c \) is the wave speed. Using the change of variables \( \xi = x - ct \) and \( \eta = x + ct \) show that the wave equation can be written in canonical form
\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \]

Hence deduce d’Alembert’s solution
\[ u = f(\xi) + g(\eta), \]  
(2)

where \( f \) and \( g \) are arbitrary functions.

Verify by direction substitution that (2) is a solution (1).²

THE END

²In fact (2) is the general solution of (1) for the arbitrary functions \( f \) and \( g \).