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Scaling degree and extensions of distributions

We have an action of $\mathbb{R}_+$ on test functions $D(\mathbb{R}^d)$

$$\mathbb{R}_+ \times D(\mathbb{R}^d) \to D(\mathbb{R}^d).$$

$$(\lambda, \varphi) \to \varphi^\lambda(x) = \lambda^{-d} \varphi(\lambda^{-1} x)$$

we set for $t \in D'(\mathbb{R}^d)$ $\quad t_\lambda(\varphi) = t(\varphi^\lambda)$

if $t \in L^1_{loc}(\mathbb{R}^d)$ $\quad t_\lambda(\varphi) = \int t(\lambda x)\, \varphi(x)\, dx$

Def: t has scaling degree $sd(t) = \omega$ at the origin if ω is the inf of ω' s.t.

$$\lim_{\lambda \to 0^+} \lambda^{\omega'} t_\lambda = 0 \quad \text{as dist.}$$

Ex ① if $t_\lambda = \lambda^\alpha t$ $\quad sd(t) = -\alpha$

② $\mu(\varphi) = \varphi(0)$ $\quad sd(\mu) = d$.

③ Feynman Propagator. $E_F(x) = (2\pi)^{-d} \int \frac{e^{ip\cdot x}}{p^2 - m^2 + i\varepsilon}\, dp$

$sd(E_F) = d-2$

Remark: every t has $sd(t) \in [-\infty, \infty)$

<u>Lemma</u> Let $t \in D'(\mathbb{R}^d)$, $sd(t) = \omega$ at 0.

$sd(\partial^\alpha t) \leq \omega + |\alpha|$
$sd(x^\alpha t) \leq \omega - |\alpha|$

$sd(ft) \leq sd(t)$

$sd(t_1 \otimes t_2) = sd(t_1) + sd(t_2)$

<u>Thm</u> Let $t_0 \in D'(\mathbb{R}^d \setminus \{0\})$ has scaling degree $\omega = sd(t_0) < d$

$\exists!\, t \in D'(\mathbb{R}^d)$ with scaling degree ω s.t.
$t(\varphi) = t_0(\varphi)$ if $\operatorname{supp} \varphi \not\ni \{0\}$

<u>Proof</u>: <u>unique</u>: the diff of two extensions will be supported at the origin has the form $P(\partial)\delta$ So degree of P must be 0.

<u>exists</u> Let Θ be a smooth function st. $\Theta = 1$ near the origin $\Theta_\lambda(x) := \Theta(\lambda x)$

$t^{(n)} = (1 - \Theta_{2^n}) t_0$ claim it is a cauchy seq

let $\rho = \omega - d$ (degree of singularity)
Let $D_\rho(\mathbb{R}^d)$ be the space of smooth functions of compact support that vanish to order ρ at 0. Let W be a projection to this subspace.

$$W\varphi = \varphi - \sum_{|\alpha| \leq \rho} m_\alpha \partial^\alpha \varphi(0)$$

$m_\alpha \in D(\mathbb{R}^d)$ s.t. $\partial^\alpha m_\beta = \delta^\alpha_\beta$

Thm Let $t_0 \in D'(\mathbb{R}^d \setminus \{0\})$ have a finite Scaling deg $\omega \geq d$ exits extensions $t \in D'(\mathbb{R}^d)$ with the same scaling given W they are uniquely determined by their values on m_α

Proof any $\varphi \in D(\mathbb{R}^d)$ can be uniquely ~~determined~~ ~~by~~ decomposed $\varphi = \varphi_1 + \varphi_2$

where $\varphi_1 = \sum_{|\alpha| \leq \rho} m_\alpha \partial^\alpha \varphi(0)$

$\varphi_2 \in D_\rho(\mathbb{R}^d)$

$\varphi_2 = \sum_{|\alpha| = [\rho]+1} x^\alpha \psi_\alpha(x) \quad \psi_\alpha \in D(\mathbb{R}^d)$

Let $\langle t, \varphi \rangle = \sum_{|\alpha| \leq [\rho]+1} \langle x^\alpha t_0, \psi_\alpha \rangle + \langle t_0, \varphi_1 \rangle$

Since $x^\alpha t_0$ has sd, $\rho - [\rho] - 1 + d < d$ it has a unique extension to 0.

check: $sd(t) = sd(t_0)$

Thm: Let t_0 be an Lorentz inv. tensor valued dist on $\mathbb{R}^{4n}\setminus\{0\}$. s.t $\exists N$ $E_\rho^N t_0 = 0$ $\quad$ $E_\rho = E + \rho$. (Euler operator)

Then t_0 has a Lorentz inv. extension t to $\mathbb{R}^{4n}$ with the same homog. $\exists N$ $E_\rho^N t = 0$.

Proof. $$Wf = f - w \sum_{|\alpha|\le \rho - 4n} y^\alpha \frac{\partial_\alpha f(0)}{\alpha!}$$

from $E_\rho^N t_0 = 0$ it follows that $\operatorname{sd}(t_0) = \rho$.
So we can extend t_0 by $t(f) = t_0(Wf)$.
Need an extension s.t. $E_\rho^N t = 0$ for some N.

$$W E_\rho^N f - E_\rho^N W f = \sum_{|\alpha|\le \rho-4n} \varphi^\alpha \partial_\alpha f(0) \qquad \operatorname{supp}\{\varphi^\alpha\} \not\ni 0$$

$$E_\rho^N t = \sum_{|\alpha|\le\rho-4n} c^\alpha \partial^\alpha \delta \qquad c^\alpha = (-1)^{|\alpha|} t^0(\varphi^\alpha)$$

$$t' = t - \sum_{|\alpha|\le \rho-4n} \frac{c^\alpha}{(\rho-4n-|\alpha|)^N} \partial^\alpha \delta.$$

since $E_\rho \partial^\alpha \delta = (\rho - 4n - |\alpha|)\partial^\alpha\delta$

we get $$E_\rho^N t = \sum_{|\alpha|\le\rho-4n} c^\alpha \partial^\alpha \delta$$

so $E_\rho^{N+1} t' = 0$ $\quad$ since $E_\rho \partial^\alpha \delta = 0$ for $|\alpha| = \rho - 4n$.

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~~W~~ Lorentz inv extensions

W

$u = u_0 \circ W$ $\quad u_{\mu_1 \cdots \mu_\ell}(f) - \Lambda_{\mu_1}^{\nu_1} \cdots \Lambda_{\mu_\ell}^{\nu_\ell} u_{\nu_1 \cdots \nu_\ell}(R(\Lambda) f)$

$$= \sum_{|\alpha| \leq \ell - 4n} b^\alpha_{\mu_1 \cdots \mu_\ell} \partial_\alpha \delta(f)$$

for some constants $b^\alpha_{\mu_1 \cdots \mu_\ell}$

apply E_ℓ^{N+1} to both sides (Note that E_ℓ is Lorentz inv)

$$0 = E_\ell^{N+1} \sum_{|\alpha| \leq \ell - 4n} b^\alpha_{\mu_1 \cdots \mu_\ell} \partial_\alpha \delta = \sum_{|\alpha| \leq \ell - 4n} (\ell - 4n - |\alpha|)^{N+1} b^\alpha_{\mu_1 \cdots \mu_\ell} \partial_\alpha \delta$$

$\Rightarrow b^\alpha_{\mu_1 \cdots \mu_\ell}(\Lambda) = 0$ except possibly for $|\alpha| = \ell - 4n$.

b is valued in $(\otimes^\ell \mathbb{R}^4)^* \otimes (\otimes^{\ell - 4n} \mathbb{R}^4) \leftarrow D$ is this rep.

Check

$\Rightarrow b(\Lambda_1 \Lambda_2) = b(\Lambda_1) + D(\Lambda_1) b(\Lambda_2)$

So b is a 1-cocycle for the Lorentz group valued in this Rep. Defines element in $H^1(L, \cdots) = 0$.

get that $b(\Lambda) = a - D(\Lambda) a$

a is a vector in the rep.

Set $u'_{\mu_1 \cdots \mu_\ell} = u_{\mu_1 \cdots \mu_\ell} - a^{\nu_1 \cdots \nu_{\ell - 4n}}_{\mu_1 \cdots \mu_\ell} \partial_{\nu_1} \cdots \partial_{\nu_{\ell - 4n}} \delta$

Check that u' is Lorentz invariant.

Renormalizability

dimension d $\quad S = T(\exp \varphi^k)$

$T_n(\varphi^k \cdots \varphi^k) = \sum t : \cdots :$

~~$T_n \varphi = \sum T_i T_{n-i}$~~

$T_n = \sum f_I \, T_i \, T_{n-i}$

Claim: The dist. in the expansion of T_n° are of the form $f_I t^I(x_I) t^{I^c}(x_{I^c}) \prod_{\substack{i \in I \\ j \in I^c}} \omega_2(x_i, x_j)^{a_{ij}}$ $\quad (*)$

$\{a_{ij} \in \mathbb{N}\}$

$sd_{\Delta_2} \omega_2 = d-2 \quad (d = \dim M)$

Assume: f_I have zero sd at Δ_n

we get that sd of $*$ is bounded by

$sd_{\Delta_I}(t^I) + sd_{\Delta_{I^c}}(t^{I^c}) + \sum_{ij} a_{ij}(d-2)$

by induction we get $sd_{\Delta_n}(\omega T(\otimes \varphi^{l_i}(x_i)) \leq \sum_{i=1}^{n} l_i \frac{d-2}{2}$

$d=4 \quad \varphi^4 \quad sd_{\Delta_n} \leq 4n \frac{4-2}{2} = 4n$

$\text{codim}(\Delta_n, M^n) = 4n-4$

difference $sd_{\Delta_n} - \text{codim}(\Delta_n, M^n) = 4$ for any n.

so φ^4 on $d=4$ is renormalizable.

$sd_N t < \text{codim}_N M \Rightarrow \exists!$ extension

$\infty > sd_N t > \text{codim}_N M \Rightarrow \exists$ extension.

$d=4 \quad \varphi^N \quad sd = nN$

$\text{codim} = 4n-4 \quad \text{diff} = 4 + (n-4)N$

$d=6 \quad sd = 2nN \quad n>3$ non renorm.

$\text{codim} = 6n-6$

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finish the construction:

We want to define T_n. By expansion axiom, need to define some scalar-valued dist t_n. By factorization we know t_n^0 (defined on $M^n \setminus \Delta_n$)

Scaling expansion (follows from continuity & microlocal cond)

$$t_0 = \sum_{k=0}^{m-1} \frac{1}{k!} \tau_k^0 + r_m^0$$

$$\tau_k^0(x,y) = \sum C(x) \, \alpha_u^* u^0(y)$$

where u^0 are Lorentz inv. dist on $\mathbb{R}^{4n} \setminus \{0\}$

By extending the u^0's to all of $\mathbb{R}^{4n}$ in a Lorentz inv way and preserving the scaling degree we can extend τ_k^0.

by choosing m big enough $sd_{\Delta_n}(r_m^0) \leq 4n-1$ so it has a unique extension r_m.

So we can define $t = \sum_{k=0}^{m-1} \tau_k + r_m$

Claim: These extension satisfy all the axioms. So Time ordered products exist! $\square$

Renormalization ambiguities:

Lemma: Suppose we have T_1 and $\tilde{T}_1$ two prescription for normal ordering $T_1(\varphi^k) = \varphi^k$. $\tilde{T}_1(\varphi^k) = \tilde{\varphi}^k$

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}(x) \varphi^i(x)$$

$C_k(x) = C_k(g_{ab}(x), R_{abcd}(x), \dots, \nabla_{e_1} \cdots \nabla_{e_k} R_{abcd}, m^2, \xi)$

polynomials with real coeffs analytic in ξ.

$C_k \to \lambda^k C_k$ when $g_{ab} \to \lambda^{-2} g_{ab}$, $m^2 \to \lambda^2 m^2$, $\chi \to \chi$.

<u>Proof</u>: induction $\tilde{\varphi}^1 = \varphi^1$ by one of the axioms.
Suppose we defined the C_i for $i \leq k$.
look at $\phi_k = \tilde{\varphi}^k - \varphi^k - \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}\, \varphi^i(x)$

<u>Claim</u>: $[\phi_k(x), \varphi(y)] = 0$ follows from induction and expansion axiom.

$$[\varphi^k(x), \varphi(y)] = k\varphi^{k-1} G(x,y)$$

hence $\phi_k(x)$ is a scalar dist of the form $C_k(x) \cdot 1$. using the other axioms, we get that $C_k(x)$ has the desired form.

$\tilde{\varphi}^2 = {:}\varphi^2{:}_H + Z m^2$ will also work.

<u>Recall</u>: $S(L) = T(\exp i\lambda L)$ $\quad S(L) \in W(M,g)[[\lambda]]$

<u>Thm</u>: let T and $\tilde{T}$ be two prescriptions for time ordering. Then $\tilde{S}(\phi) = S(\phi + \delta(\phi))$
where $\delta(\phi) = \sum_{n\geq 1} \frac{\lambda^{n-1}}{n!} O_n(\phi^{\otimes n})$

where $O_n : \otimes^n \mathcal{D}(M,V) \to \mathcal{D}(M,V)$
with supp in Δ_n $\quad O_n(\otimes_{i=1}^n f_i \phi_i) = \sum F_j \varphi_j$
φ_j is a basis for V
$F_j(x) = \sum C_j^{(a)}(x) \prod \nabla_{(a_i)} f_i(x)$
These O_n's satisfy extra requirements. (…)

<u>Ex</u>: find the scaling dimension requirement on O_n, and deduce again which theories are renormalizable where.

Def: we call L renormalizable $L = \sum f_i \phi_i$
if $\tilde{S}(L) = S(\tilde{L})$ when $\tilde{L} = \sum \tilde{f}_i \phi_i$.

5/12/08 $L = \varphi^4$ $L e^{\gamma}$
QFT fL, f a cutoff

we want to take the limit $f \to 1$.
Adiabatic limit. $\Rightarrow$ infrared divergences.
Recall $S(\phi) = T(\exp(i\lambda\varphi))$

$$= 1 + \sum \frac{(i\lambda)^n}{n!} T_n(\varphi^{\otimes n})$$

claim $S(A+B+C) = S(A+B)S(B)^{-1}S(B+C)$
if supp(A) is later than supp(C).
relative S-matrix
$S_{gL}(A) = S(gL)^{-1}S(gL+A)$ due to Bogoliubov-Shirkov

Claim: If Supp(A) and Supp(B) are space like sep
then $S_{gL}(A+B) = S_{gL}(A)S_{gL}(B) = S_{gL}(B)S_{gL}(A)$

$S_{gL}(A)$ are the local observables.
Prop: Let $G \subset M$ be a causally closed manifold and g, g' test functions that coincide on a neighborhood of G. $\exists$ a unitary V s.t. for all A,
$V S_{gL}(A) V^{-1} = S_{g'L}(A)$ $(\operatorname{supp} A \subset O)$

Proof $g' - g = a + b$ where $\text{supp}(a)$ doesn't intersect the past of $\mathcal{O}$ and $\text{supp}(b)$ doesn't intersect the future of $\mathcal{O}$.

$\text{supp}(aL)$ is later than $\text{supp}(A)$

$$S(g'L + A) = S(g'L)\, S((g+b)L)^{-1}\, S((g+b)L + A)$$

$$\Rightarrow S_{g'L}(A) = S_{(g+b)L}(A)$$

Also support(A) is later than $\text{supp}(bL)$

so $\quad S((g+b)L + A) = S(gL + A)\, S(gL)^{-1}\, S((g+b)L)$

So we can choose V to be $V = S_{gL}(bL)^{-1}$

If we want to define the alg of observables supp on $\mathcal{O}$ we can take

$$\lim \{\text{alg gen by } S_{gL}(A)\}$$

$\text{supp}\, A \subset \mathcal{O}$

g as before

Lemma: Let (M, g) be a glob hyp spacetime

$\exists$ seq of compact sets $\{K_n\}$

s.t. ① $K_n \subset V_{n+1}$ $\quad V_{n+1} = \text{int}(K_{n+1})$

② $\cup K_n = M$

③ V_n is a glob hyp and $\Sigma \cap V_n$ is Cauchy surf for V_n where Σ is Cauchy surf for M.

Let $\{K_n\}$ be as in the Lemma.
and $\{\Theta_n\}$ test functions s.t. $\mathrm{Supp}\,\Theta_n \subset K_{n+1}$

and $\Theta_n = 1$ near K_n.
Let $\mathcal{U}_n = \mathcal{U}(\Theta_n, \Theta_{n-1})$ be the unitary

op defined in the prop.

let $u_n = \mathcal{U}_1 \mathcal{U}_2 \cdots \mathcal{U}_n$
Define $S_L(\phi) = \lim_{n\to\infty} u_n S_{\Theta_n L}(\phi) u_n^{-1}$

Define The interacting time ordered product as
$$S_L(\Sigma f_i \phi_i) = 1 + \sum \frac{(i\lambda)^n}{n!} T_L\left((\Sigma f_i \phi_i)^{\otimes n}\right)$$
$$T_L(\otimes f_i \phi_i) := \frac{\partial^n}{i^n \partial\alpha_1 \cdots \partial\alpha_n} S_L(\Sigma \alpha_i f_i \phi_i)\Big|_{\alpha_i = 0}.$$

Def $B_L(M,g) := \{\text{alg gen by } T_L(\otimes f_i \phi_i)\}$
$\quad\quad \Psi$
$T_L(\varphi^n)$ interacting wick monomials.

Claim: If $\{K_n\}$ and $\{K_n'\}$ are two choices of compact sets as above we can construct
$\gamma : B_L(M,g) \xrightarrow{\sim} B_L(M,g)$

Claim $(M,g) \to B_L(M,g)$ is a QFT. and $T_L(\cdots)$ are quantum fields.