4 Gelfand-Naimark-Segal Construction of Representations

Let A be a C^*-algebra, and let $\phi : C^* \to \mathbb{C}$ be a state. We can define a bilinear form,

$$<,>_\phi: A \otimes \bar{A} \to \mathbb{C}$$
$$< a, b >_\phi = \phi(b^* a)$$

Then $<,>$ is a positive semi-definite Hermitian form, i.e.

1. $< a, b >_\phi = \overline{< b, a >}_\phi$
2. $< b, b >_\phi \geq 0, \forall b$ (Note this can be zero).
3. $| < a, b >_\phi |^2 \leq < a, a >_\phi < b, b >_\phi$

Definition 1. A representation of a C^*-algebra on a Hilbert space H is a map $\pi : A \to B(H)$ of C^*-algebras.

Theorem 4.1. *To every state phi, there corresponds a Hilbert space H_ϕ, a representation π_ϕ on H_ϕ, and a vector $v \in H_\phi$ s.t.*

1. $\phi(a) = < \pi_\phi(a)v, v >$.
2. *v is cyclic for H_ϕ (i.e. $\pi_\phi(A)v$ is dense)*

Moreover, H_ϕ is unique up to unitary equivalence.

Proof. We demonstrate existence, as uniqueness is clear by the cyclicity assumption. Let $K_\phi = \{a \in A | \phi(a^* a) = 0\}$. From Cauchy's inequality, we can see that $K_\phi = \{a \in A | \phi(b^* a) = 0 \forall b \in A\}$. Hence K_ϕ is a closed left ideal. A/K_ϕ is thus a left A-module, with a pre-Hilbert space structure. We define H_ϕ as the completion of A/K_ϕ, and we let v denote the image of $1 \in A$ inside H_ϕ. □

Definition 2. Let $C \subset D$ be rings. The commutant, C', denotes all the elements of D which commute with those of C. C'' denotes the commutant of the commutant of C.

Definition 3. A representation H is called irreducible if any of the three equivalent statments hold:

1. $0, H$ are the only A-stable subspaces.
2. $\pi(A)'' = B(H)$.
3. $\phi(A)' = \mathbb{C}.id$.

Claim 4.2. *π_ϕ is irreducible if, and only if, ϕ is a pure state.*

5 States on a commutative C^* algebra

Suppose A is a commutative C^* algebra, and $\phi \in A^*$.

Claim 5.1. *The following are equivalent:*

1. *ϕ is a pure state.*
2. *ϕ is multiplicative: $\phi(ab) = \phi(a)\phi(b), \forall a, b \in A$.*

Proof. Let ϕ be a state. Then ϕ is pure $\Leftrightarrow H_\phi$ is irreducible $\Leftrightarrow H_\phi$ is 1-dimensional $\Leftrightarrow \phi$ is multiplicative. Note the middle implication is where we used commutativity of A. $\square$

Definition 4. Let A be a commutative C^* algebra. A character on A is a multiplicative, non-zero linear map $\phi : A \to \mathbb{C}$. We let $\mathrm{Spec}(A)$ denote the set of all characters, which by the previous claim is in bijection with the set of all pure states.

Theorem 5.2. *Let A be a commutative C^*-algebra, and let $X = Spec(A)$ with the weak $*$-topology generated by neighborhoods*

$$N_{\phi,a_1,\ldots,a_n}(\epsilon) = \{\phi' || \phi(a_i) - \phi'(a_i) < \epsilon\}$$

.

Then X is a locally compact Hausdorff space, and $A \cong C_0(X)$, the closure of compactly supported functions on X, via the correspondence $a \mapsto \hat{a}, \hat{a}(\phi) = \phi(a)$. X is compact if, and only if, A is unital.

Theorem 5.3. *The category of locally compact Hausdorff spaces is equivalent to the opposite category of commutative C^* algebras.*

Corollary 5.4. *$\{$Mixed states on $A\}$ are in bijection with $\{$probability measures on $X\}$.*

6 Quantization

Classical mechanics takes place on a phase space X, which is a symplectic manifold. $C_0(X)$ is thus a Poisson algebra, i.e. we have a Lie bracket $\{,\} : C_0(X) \otimes C_0(X) \to C_0(X)$ which is a bi-derivation. To $\{,\}$ corresponds an antisymmetric bivector field B on X given by $B(df, dg) = \{f, g\}$.

Definition 5. Let A_0 be a commutative C^* algebra with a Poisson structure. A quantization of A_0 is the data of an interval I containing 0, a C^* algebra $A_\hbar$ for each $\hbar \in I$, and linear maps $Q_\hbar : A_0 \to A_\hbar$, s.t.

1. $Q_0 = id$.
2. $\forall f$, $\hbar \mapsto ||Q_\hbar(f)||$ is constant.
3. $\lim_{\hbar\to 0} ||Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)|| = 0$.
4. $\lim_{\hbar\to 0} ||[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}|| = 0$.
5. For each $\hbar \in I$, $Q_h(A_0)$ is dense in $A_\hbar$.

We call it strict if $Q_\hbar(A_0)$ is closed under multiplication, and nondegenerate if $\mathrm{Ker} Q_\hbar = 0$.