Homework Assignment 3

Due Friday, October 10.

Exercise 1. (The Borel theorem). Show that for every formal power series in n variables:

$$\sum a_\alpha x^\alpha,\ 0 \le |\alpha| < \infty \tag{1}$$

there exists a function, f, in $C^\infty(\mathbb{R}^n)$ whose Taylor series at $x = 0$ is the series (1).

Hint. Let $\rho(t) \in C_c^\infty(\mathbb{R})$ be a function which is 1 for $|t| \le \frac{1}{2}$ and 0 for $|t| > 1$, and let c_m be the supremum over $|x| \le 1$ of the sum

$$\sum_{|\alpha| \le m} |\partial^\alpha \rho(|x|)|$$

and a_m the supremum over $|\alpha| = m$ of $|a_\alpha|$. Show that for

$$0 < \epsilon_m \le \frac{1}{(m+1)c_m(a_m+1)}$$

the series

$$f(x) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} a_\alpha x^\alpha \rho\left(\frac{|x|}{\epsilon_m}\right)$$

is a C^∞ function and that its Taylor series at $x = 0$ is the series (1). (**Subhint**. Show that for $x \ne 0$ all but finitely many terms in the series above are zero.)

Exercise 2. Show that the function, $f : \mathbb{C} \to \mathbb{C}, f(z) = \frac{1}{z}$, is a locally L^1 function and that it satisfies the distributional equation

$$\frac{1}{\pi} \frac{\partial}{\partial \bar{z}} f = \delta_0(z).$$

Additional Exercises (in Frielander-Joshi): page 31, exercises 2.8 and 2.9, and page 39, exercise 3.1.