Homework Assignment 2

Due Friday, September 26.

In the applications of distribution theory that we'll be considering in the next few weeks we'll need a slightly stronger version of the partition of unity theorem (**theorem 1.4.1**) in Friedlander, namely we'll need the following.

Theorem. *Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$ and $\{\mathcal{U}_\alpha, \alpha \in \mathcal{I}\}$ a covering of $\mathcal{U}$ by open sets. Then there exists a sequence of functions, $\psi_i \in C_c^\infty(\mathcal{U}), i = 1, 2, \ldots,$ such that*

1. $\psi_i \geq 0$
2. *For every i, ψ_i is supported in $\mathcal{U}_\alpha$ for some α*
3. *If K is a compact subset of $\mathcal{U}, \operatorname{supp} \psi_i$ and K are disjoint for all but a finite number of i's*
4. $\sum \psi_i = 1$

Remark. *Note that the sum in 4 makes sense since, by 3, this sum is finite on every compact subset of $\mathcal{U}$.*

Exercise 1. Show that there exists a sequence of compact subsets, A_k of $\mathcal{U}$ such that A_k is contained in the interior of A_{k+1} and such that the union of the A_k's is $\mathcal{U}$.

Hint. Try defining A_k as the intersection of the ball $|x| \leq k$, with the set of points, x, in $\mathcal{U}$ which are a distance less than or equal to $1/k$ from the complement of $\mathcal{U}$ in $\mathbb{R}^n$, i.e. which satisfy

$$\inf\ (|x - y|, y \in \mathcal{U}^c) \leq 1/k$$

Exercise 2. Using **theorem 1.4.1** show that there exist functions

$$\psi_{i,k}\ ,\ 1 \leq i \leq N_k$$

with the properties 1 and 2 of the theorem above and in addition, the properties

3′. For every i the support of $\psi_{i,k}$ is contained in the open set $\text{Int}\, A_{k+2} - A_{k-1}$.

4′. $\sum_i \psi_{i,k} = 1$ on the compact set, $A_{k+1} - \text{Int}\, A_k$.

Exercise 3. Let $\psi_1, \psi_2, \psi_3, \ldots$ be a relabeling of the sequences

$$\psi_{1,1}, \ldots, \psi_{N_1}, \psi_{1,2}, \ldots$$

Show that this sequence has the properties 1, 2 and 3 of the theorem above and in place of property 4 the property:

$$\psi = \sum \psi_i \geq 1$$

Exercise 4. Show that the ψ_i's can be converted into a sequence having property 4 as well as by replacing ψ_i by ψ_i/ψ

Additional Exercises. Do exercises **1.1** through **1.5** in chapter 1 of Friedlander (paying special attention to exercise **1.3**).