Homework Assignment 1

Due 9/12/14.

(The problems below have to do with weak solutions of d'Alembert's equation)

$$\text{(I)} \qquad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0$$

i.e. continuous functions, $u(x,t) \in C^0(\mathbb{R}^2)$ which satisfy

$$\text{(II)} \qquad \int u(x,t) \left(\frac{\partial^2}{\partial t^2} d - \frac{\partial^2}{\partial x^2} d\right) \, dx \, dt$$

for all functions $d \in C_c^\infty(\mathbb{R}^2)$.

Exercise 1. Show that there exists weak solutions which are not C^2. More explicitly, show that if f and g are in $C(\mathbb{R})$ the function

$$\text{(III)} \qquad u(x,t) = f(x+t) - g(x-t)$$

is such a solution.

Exercise 2. (This is harder.) Show that every weak solution is of this form.

One method for doing so is outlined in the following exercises.

Exercise 3. Show that for the simple equation in one variable

$$\frac{du(s)}{ds} = 0$$

the only weak solutions, $u \in C(\mathbb{R})$, satisfying this equation are the constants.

Exercise 4. Show that for the equation in two variables

$$\text{(IV)} \qquad \frac{\partial^2}{\partial s\,\partial t}\,u(s,t) = 0$$

every weak solution, $u \in C^0(\mathbb{R}^2)$, is of the form $f(s) + g(t)$ with f and g in $C(\mathbb{R})$.

Hint. $u(s,t)$ is a weak solution of this equation if

$$\int u(s,t)\,\frac{\partial^2}{\partial s\,\partial t}\,\phi(s,t)\;ds\,dt = 0$$

for all $\phi \in C_c^\infty(\mathbb{R}^2)$. Let $\phi(s,t) = \psi(s)\rho(t)$ and, by integration by parts, rewrite this equation as

$$0 = \int \frac{\partial\psi}{\partial s}\left(\int \frac{\partial\rho}{\partial t}\,u(s,t)\,dt\right)ds$$

to conclude that the inner integral is a constant function of s and, in particular, that for $g(t) = u(0,t)$

$$0 = \int \frac{\partial\rho}{\partial t}\,(u(s,t) - g(t))\,dt$$

for all $\rho(t)$ in $C_c^\infty(\mathbb{R})$.

Exercise 5. Show that the equation (IV) can be converted into the equation (I) by a clever change of coordinates.

Exercise 6. Show that for every function, $\phi \in C^1(\mathbb{R})$, there exists a unique weak solution, $u(x,t) \in C^\perp(\mathbb{R}^2)$ of d'Alembert's equation satisfying the initial conditions

$$u(x,0) = \phi(x)$$

and

$$\frac{\partial}{\partial t}u(x,0) = 0$$

Exercise 7. In Exercise 6 show that if ϕ is supported on the interval, $-\epsilon < x < \epsilon$, then at time, $t, u(x,t)$ is supported on the union of the intervals, $t - \epsilon < x < t$, and $-t - \epsilon < x < -t + \epsilon$.