18.152 Solutions to Homework 9

(NOTE: We are supposing that $\underline{u(0,t)=0}$)

① a) By differentiating under the integral sign:

$$E'(t) = \int_0^L (\rho u_t u_{tt} + \tau u_x u_{xt})\,dx$$

$$\int_0^L u_x u_{xt}\,dx = u_x u_t \Big|_{x=0}^{x=L} - \int_0^L u_{xx} u_t\,dx$$

Since $u_x(0,t)=0$, $u_t(L,t)=0$ (the latter follows by differentiation in time), we get:

$$\int_0^L u_x u_{xt}\,dx = -\int_0^L u_{xx} u_t\,dt$$

$$\Rightarrow E'(t) = \int_0^L u_t(\rho u_{tt} - \tau u_{xx})\,dx = \gamma \int_0^L u_t u_{xxt}\,dx =$$

$$= \underbrace{\gamma u_t u_{xx} \Big|_{x=0}^{x=L}}_{=0 \text{ since } u_t(0,t)=u_t(L,t)=0.} - \gamma \int_0^L u_{xt}^2\,dx \leq 0$$

$\rightarrow$ Energy is decreasing in time.

b) If $f=g=0$, we must show that $u \equiv 0$.
This follows immediately from the fact that $E'(t) \leq 0$ since $E(0)=0 \Rightarrow E \equiv 0 \Rightarrow u \equiv 0$.

c) Look for $u(x,t) = v(x) \cdot w(t)$

$$\rho v(x) w''(t) - \tau v''(x) w(t) = \gamma v''(x) w'(t)$$

$$c^2 := \frac{\tau}{\rho},\ \varepsilon^2 := \frac{\gamma}{\rho} : \frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t) + \varepsilon^2 w'(t)} = -\lambda^2$$

$$v''(x) = -\lambda^2 v(x);\ v(0) = v(L) = 0.$$

$$\Rightarrow v_n(x) = \sin\left(\frac{n\pi}{L} \cdot x\right);\ \lambda_n = \frac{n\pi}{L}$$

On the other hand:

$$w'' + \varepsilon^2 \lambda_n^2 w' + c^2 \lambda_n^2 w = 0$$

Consider the discriminant $D_n = \varepsilon^4 \lambda_n^4 - 4c^2\lambda_n^2$.

- If $D_n < 0$,

$$w_n(t) = \exp\left(-\frac{\varepsilon^2\lambda_n^2}{2}t\right)\left[a_n \sin\left(\frac{\lambda_n\sqrt{|D_n|}}{2}t\right) + b_n \cos\left(\frac{\lambda_n\sqrt{|D_n|}}{2}t\right)\right]$$

- If $D_n = 0$,

$$w_n(t) = (a_n + b_n t)\exp\left(-\frac{\varepsilon^2\lambda_n^2}{2}t\right)$$

- If $D_n > 0$,

$$w_n(t) = a_n \exp\left(\frac{-\varepsilon^2\lambda_n^2 + \lambda_n\sqrt{D_n}}{2}t\right) + b_n \exp\left(\frac{-\varepsilon^2\lambda_n^2 - \lambda_n\sqrt{D_n}}{2}t\right)$$

The solution to the problem is:

$$u(x,t) = \sum_{n=1}^{+\infty} w_n(t)\sin(\lambda_n x)$$

We suppose that $f, g \in C^1(\mathbb{R})$

$$\Rightarrow \quad \sum |a_n| + \sum |b_n| < \infty.$$

- If $D_n < 0$, $|w_n(t)| \le (|a_n| + |b_n|)\exp\left(-\frac{\varepsilon^2\lambda_n^2}{2}t\right) \le (|a_n| + |b_n|) \cdot \exp\left(-\frac{\varepsilon^2\pi^2}{2L^2}t\right)$
- If $D_n = 0$, $|w_n(t)| \le (|a_n| + t|b_n|) \cdot \exp\left(-\frac{\varepsilon^2\pi^2}{2L^2}t\right)$
- If $D_n > 0$, $-\varepsilon^2\lambda_n + \lambda_n\sqrt{D_n} \le -r$ for some $r > 0$ independent of n;

(one can check $r = \frac{2c^2}{\varepsilon^2}$)

$$|w_n(t)| \le |a_n|\exp\left(-\frac{2c^2}{\varepsilon^2}t\right) + |b_n|\exp\left(-\frac{\varepsilon^2\pi^2}{2L^2}t\right)$$

It follows that $u(x,t) \to 0$ exponentially fast.

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2) Done on the Practice Final Exam)

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3) $D = t^2 - t^2 = 0 \Rightarrow$ Equation is <u>Parabolic</u>.

$$t^2 u_{tt} + 2t u_{xt} + u_{xx} = (t\partial_t + \partial_x)^2 u$$

$$(t\partial_t + \partial_x)\varphi = 0$$

$$\Rightarrow \underline{\varphi(x,t) = g(t e^{-x})}$$

$\underline{t e^{-x} = k}$ is a family of characteristics.

We change variables by:

$$\begin{cases} \xi = te^{-x} \\ \eta = \psi(x) \end{cases}$$

ψ is an invertible function which we will determine later.

$$u(x,t) = \mathcal{U}(te^{-x}, \psi(x))$$

$$\Rightarrow \begin{cases} u_x = -te^{-x}\mathcal{U}_\xi + \psi'\mathcal{U}_\eta \\ u_t = e^{-x}\mathcal{U}_\xi \\ u_{xx} = te^{-x}\mathcal{U}_\xi + t^2e^{-2x}\mathcal{U}_{\xi\xi} - 2\psi' te^{-x}\mathcal{U}_{\xi\eta} + (\psi')^2\mathcal{U}_{\eta\eta} + \psi''\mathcal{U}_\eta \\ u_{xt} = -e^{-x}\mathcal{U}_\xi - te^{-2x}\mathcal{U}_{\xi\xi} + e^{-x}\psi'\mathcal{U}_{\xi\eta} \\ u_{tt} = e^{-2x}\mathcal{U}_{\xi\xi} \end{cases}$$

$$\Rightarrow (\psi')^2\mathcal{U}_{\eta\eta} + (\psi'' - \psi)\mathcal{U}_\eta = 0$$

Let us take $\psi(x) = e^x$. Then, we obtain:

$$\mathcal{U}_{\eta\eta} = 0$$

$$\Rightarrow \mathcal{U}(\xi,\eta) = F(\xi) + G(\xi)\eta$$

$$\Rightarrow \boxed{u(x,t) = F(te^{-x}) + G(te^{-x})\cdot e^x}$$

④ Recall Kirchoff's Formula (Theorem 5.4. on Page 277):

$$u(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} g(\sigma)d\sigma\right] + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} h(\sigma)d\sigma$$

We observe that $u(x,t)=0$ unless there exists $y \in B_\rho(0)$ such that $|x-y| = ct$.

Then, by the triangle inequality:

$$|x| \leq |x-y| + |y| \leq ct+\rho$$

$$|x| \geq |x-y| - |y| \geq ct-\rho$$

So $\operatorname{supp} u(\cdot,t) \subseteq \overline{B_{\rho+ct}(0)} \setminus B_{ct-\rho}(0)$.

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⑤ $\begin{cases} w_{tt} - c^2\Delta w = 0 & ; x\in\mathbb{R}^3, t>0 \\ w(x,0)=0,\ w_t(x,0) = h(|x|) & ; x\in\mathbb{R}^3 \end{cases}$

$$h(r) = \begin{cases} 1 : & 0\leq r\leq 1 \\ 0 : & r>0 \end{cases}$$

We want to show that w displays a discontinuity at $t=\frac{1}{c}$.

- We know that $w(x,0)$, $w_t(x,0)$ are radial, so w will be radial: $w = w(r,t)$

$$w_{tt} - c^2\Delta w = w_{tt} - c^2\left(w_{rr} + \frac{2}{r}w_r\right) = 0 =$$

$$= w_{tt} - c^2 \cdot \frac{1}{r}\cdot (rw)_{rr} = 0$$

$$\Rightarrow (rw)_{tt} - c^2(rw)_{rr} = 0$$

Let $v := rw$. Then v solves the 1D Wave Equation.

$$v(r,t) = F(r+ct) + G(r-ct)$$

$$\Rightarrow w(r,t) = \frac{F(r+ct)}{r} + \frac{G(r-ct)}{r}$$

$w(x,0)=0 \Rightarrow F(r)=-G(r).$

$$w_t(x,0) = \frac{cF'(r)-cG'(r)}{r} = h(r)$$

$$\frac{-2cG'(r)}{r} = h(r)$$

$$G'(r) = -\frac{r}{2c}\cdot h(r)$$

$$\Rightarrow \boxed{G(r) = -\frac{1}{2c}\int_0^r s\,h(s)\,ds}$$

Now: $w(r,t) = -\dfrac{G(r+ct)}{r} + \dfrac{G(r-ct)}{r} =$

$$= \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,h(s)\,ds$$

· If $-1 < r-ct < r+ct < 1$, $r \neq 0$ ... in particular $t < \frac{1}{c}$

$$w(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,ds = \frac{1}{2cr}\,\frac{1}{2}s^2\Big|_{r-ct}^{r+ct} = t$$

We let $r \searrow 0$ to deduce $w(0,t)=t$ for $t < \frac{1}{c}$.

Consequently, $w(0,t) \to \frac{1}{c}$, as $t \to \frac{1}{c}-$

On the other hand, if $r-tc < -1$, $r+tc > 1$,

$$w(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,h(s)\,ds = \frac{1}{2cr}\int_{-1}^{1} s\,ds = 0$$

So, we can deduce that w is discontinuous at $(0, \frac{1}{c})$.

⑥ We use <u>Duhamel's method</u>. The key is to first solve, for $0 \le s < t$:

$$\begin{cases} w_t + v w_x = 0 \\ w(x, s; s) = f(x, s) \end{cases}$$

By the method of characteristics:

$$w(x, t; s) = f(x - v(t-s), s)$$

Duhamel's method tells us that:

$$u(x,t) = \int_0^t w(x,t;s)\,ds =$$

$$= \int_0^t f(x - v(t-s), s)\,ds$$

When $f(x,t) = e^{-t}\sin x$, it follows that:

$$u(x,t) = \int_0^t e^{-s}\sin(x - v(t-s))\,ds =$$

$$= \operatorname{Im} \int_0^t e^{-s} e^{i(x-vt+vs)}\,ds =$$

$$= \operatorname{Im}\left\{ e^{i(x-vt)} \int_0^t e^{(-1+iv)s}\,ds \right\} =$$

$$= \operatorname{Im}\left\{ e^{i(x-vt)} \cdot \frac{1}{-1+iv} \cdot \left(e^{(-1+iv)t} - 1\right) \right\} =$$

$$= \operatorname{Im}\left\{ e^{i(x-vt)} \cdot \frac{-1-iv}{1+v^2} \cdot \left(e^{(-1+iv)t} - 1\right) \right\} =$$

$$= \frac{1}{1+v^2}\operatorname{Im}\left\{ e^{ix-t}\cdot(-1-iv) + e^{i(x-vt)}\cdot(1+iv) \right\} =$$

$$= -\frac{e^{-t}}{1+v^2}\cdot\sin x - \frac{ve^{-t}}{1+v^2}\cos x + \frac{1}{1+v^2}\sin(x-vt) + \frac{v}{1+v^2}\cos(x-vt)$$

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7. $\frac{d}{dt}\int_0^R u^2(x,t)\,dx = \int_0^R 2u u_t(x,t)\,dx =$

$= \int_0^R 2u f(x,t)\,dx - \int_0^R 2u a u_x(x,t)$

$\int_0^R u u_x(x,t)dx = \frac{1}{2}\int_0^R \frac{\partial}{\partial x}(u^2)(x,t)\,dx = \frac{1}{2}u^2(R,t) - \frac{1}{2}u^2(0,t) =$

$= \frac{1}{2}u^2(R,t) \geq 0$

$\Rightarrow \frac{d}{dt}\int_0^R u^2(x,t)dx \leq \int_0^R 2u(x,t)\, f(x,t)\,dx$

$\leq \int_0^R u^2(x,t)\,dx + \int_0^R f^2(x,t)\,dx$

The claim follows if we show:

$(*)$ If $E(t)$ satisfies:

$E'(t) \leq G(t) + E(t)$: $E(0)=0$, then: $E(t) \leq e^t\{\int_0^t G(s)ds\}$

Proof of $(*)$: Let $F(t) := e^{-t}E(t)$.

We need to show that: $F(t) \leq \int_0^t G(s)ds$.

Now:

$\frac{d}{dt}F(t) = -e^{-t}E(t) + e^{-t}E'(t) \leq e^{-t}G(t) \leq G(t) = \frac{d}{dt}\left(\int_0^t G(s)ds\right)$,

since we are considering $t \geq 0$.

On the other hand $F(0) = \int_0^0 G(s)\,ds = 0$,

So, we integrate in time to deduce:

$F(t) \leq \int_0^t G(s)\,ds$, $(*)$ now follows and implies:

$\int_0^R u^2(x,t) \leq e^t \int_0^t \int_0^R f^2(x,s)\,dx\,ds$; $t>0$
