40/40 Congratulations! Excellent Work! (+10/10 for HW9)

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1 (a). Let G be the extended g as follows

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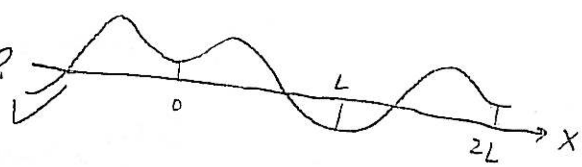
$$G(x) = g(|x-2kL|), \quad (2k-1)L \le x < (2k+1)L, \ k \in \mathbb{Z}.$$

Then G is even and periodic with period $2L$, and G is smooth at $x \neq kL$.

Since $u_x(0,t) = u_x(L,t) = 0$, $u_x(0,0) = u_x(L,0) = 0$.

Thus $g'(0) = g'(L) = 0$.

Hence $G'(kL) = 0$, in particular, $G'(0)=0$.



For $x \neq 0$, $G'(x) = \lim_{y \to x} \frac{G(y)-G(x)}{y-x}$

$$= -\lim_{y \to x} \frac{G(-y)-G(-x)}{(-y)-(-x)}$$

$$= -G'(-x).$$

$$\Rightarrow \lim_{x \to 0^+} \frac{G'(x)-G'(0)}{x} = \lim_{x \to 0^+} \frac{G'(x)}{x} = \lim_{x \to 0^+} \frac{-G'(-x)}{x} = \lim_{-x \to 0^-} \frac{G'(-x)}{-x} = \lim_{x \to 0^-} \frac{G'(x)}{x}$$

$\Rightarrow$ $G''(0)$ exists $\Rightarrow G''(0) = \lim_{x \to 0^+} \frac{G'(x)}{x} = \lim_{x \to 0^+} \frac{g'(x)}{x} = \lim_{x \to 0^+} \frac{g'(x)-g'(0)}{x-0} = g''(0)$.

Similarly, one can get $G''(x) = g''(x)$, $\forall x \in (0, L)$.

Since g is smooth, $g \in C^2$. Thus G'' is continuous $\Rightarrow G \in C^2$.

Let $U(x,t) = u(|x-2kL|, t)$, $(2k-1)L \le x < (2k+1)L$, $k \in \mathbb{Z}$.

Then
$$\begin{cases} U_{tt} - c^2 U_{xx} = 0, & x \in \mathbb{R}, \ t > 0 \\ U(x,0) = G(x), \ U_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

By d'Alambert formula, $U(x,t) = \frac{1}{2}[G(x+ct) + G(x-ct)]$, $x \in \mathbb{R}$

Thus $u(x,t) = \frac{1}{2}[G(x+ct) + G(x-ct)]$ for $x \in [0, L]$

7/7 (b) u is a superposition of two periodic and even progressive waves with period $2L$. ✓ Good!

I use the equation $u_{tt}-c^2u_{xx}=0$.

7/7 2. (a) By d'Alembert formula,

$$u(x,t)=\frac{1}{2}\left(g(x+ct)+g(x-ct)\right)$$

$$g(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$$

$$=\begin{cases}1 & -a<x-ct\le x+ct<a\\ \frac{1}{2} & \left\{\begin{array}{l}-a<x-ct<a<x+ct \text{ or } x-ct<-a<x+ct<a\end{array}\right.\\ 0 & x+ct<-a \text{ or } x-ct>a \text{ or } x-ct<-a<a<x+ct\end{cases}$$

or $-a<x+ct\le x-ct<a$ ($x-ct$, $x+ct$) ✓

7/7 (b) By d'Alembert formula,

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)\,dy$$

$$h(y)=\begin{cases}0, & |x|>a\\ 1, & |x|<a\end{cases}$$

$$=\frac{1}{2c}\left|[x-ct,x+ct]\cap[-a,a]\right|$$ ✓

$$=\begin{cases}\frac{2ct}{2c}=t, & -a<x-ct\le x+ct<a\\ \frac{2a}{2c}=\frac{a}{c}, & x-ct<-a<a<x+ct\\ (x+ct+a)/2c, & x-ct<-a<x+ct<a\\ (a-x+ct)/2c, & -a<x-ct<a<x+ct\\ 0, & x+ct<-a \text{ or } x-ct>a\end{cases}$$

18.152 Problem Set #8

.roblem 2

7.17

(a) $g(x) = H(x+a) - H(x-a)$ $\qquad h(x) = 0$

H = Heaviside Function

$u(x,t) = \frac{1}{2} g(x+ct) + \frac{1}{2} g(x-ct)$

$u(x,t) = \frac{1}{2} H(x+ct+a) - \frac{1}{2} H(x+ct-a)$

$\quad + \frac{1}{2} H(x-ct+a) - \frac{1}{2} H(x-ct-a)$

7.17

(b) $g(x) = 0$ $\qquad h(x) = H(x+a) - H(x-a)$

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\,dy = \frac{1}{2c} \int_{x-ct}^{x+ct} [H(y+a) - H(y-a)]\,dy$$

$$= \frac{1}{2c} \left[\int_{-\infty}^{x+ct} H(y+a)\,dy - \int_{-\infty}^{x-ct} H(y+a)\,dy - \int_{-\infty}^{x+ct} H(y-a)\,dy + \int_{-\infty}^{x-ct} H(y-a)\,dy \right]$$

$$= \frac{1}{2c} \Big[(x+ct+a) H(x+ct+a) - (x-ct+a) H(x-ct+a)$$

$$- (x+ct-a) H(x+ct-a) + (x-ct-a) H(x-ct-a) \Big]$$

Very Nice!

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3. (5.6) Check that formula 5.42 in text may be written as

$$u(x + c\xi - c\eta, t + \xi + \eta) - u(x + c\xi, t + \xi) - u(x - c\eta, t + \eta) + u(x, t) = 0.$$

Show that if u is C^2 function and satisfies the above, then $u_{tt} - c^2 u_{xx} = 0$.

Consider the parallelogram in Figure 5.4 in textbook, (and reproduced below), and let the point A be (x, t). Then we can write the coordinates of B as $(x_b, t + \xi)$. Then we need $x - ct = x_b - c(t + \xi)$ and observe the coordinates of B are $(x + c\xi, t + \xi)$. Similarly, if C is the point $(x_c, t + \eta)$ then $x + ct = x_c + c(t + \eta)$ and we see that C is the point $(x - c\eta, t + \eta)$. Finally, it is not difficult to see that the coordinates of D are $(x + c\xi - c\eta, t + \xi + \eta)$. Since $u(A) + u(D) = u(B) + u(C)$, we get that

$$u(x + c\xi - c\eta, t + \xi + \eta) - u(x + c\xi, t + \xi) - u(x - c\eta, t + \eta) + u(x, t) = 0.$$

Now the operator $\partial_{tt} - c^2\partial_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$. These are two directional derivatives. That is, the operator represents the directional derivative in direction $(-c, 1)$ of the directional derivative in direction $(c, 1)$. So denoting directional derivative in direction (a, b) as $\partial_{(a,b)}$, we see that

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= (\partial_{tt} - c^2\partial_{xx})u \\
&= \partial_{(-c,1)}(\partial_{(c,1)}u) \\
&= \lim_{\eta \to 0} \frac{\partial_{(c,1)}u(x - c\eta, t + \eta) - \partial_{(c,1)}u(x, t)}{\eta} \\
&= \lim_{\eta \to 0} \left[\frac{\lim_{\xi \to 0} \frac{u(x - c\eta + c\xi, t + \eta + \xi) - u(x - c\eta, t + \eta)}{\xi} - \lim_{\xi \to 0} \frac{u(x + c\xi, t + \xi) - u(x, t)}{\xi}}{\eta} \right] \\
&= \lim_{(\xi,\eta) \to (0,0)} \frac{u(x + c\xi - c\eta, t + \xi + \eta) - u(x + c\xi, t + \xi) - u(x - c\eta, t + \eta) + u(x, t)}{\xi\eta} \\
&= 0
\end{aligned}$$

Very Nice!