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4i/4i + 5 extra credit! Congratulations!

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Excellent work!

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1 (5.1)

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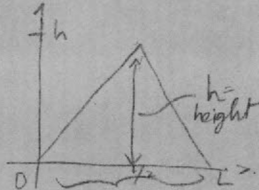
$$\begin{cases} (1) & u_{tt} - c^2 u_{xx} = 0, \quad c^2 = \frac{\tau_0}{\rho_0}, \quad 0 < x < L,\ t > 0 \\ (2) & u(x,0) = f(x) := \begin{cases} \frac{2h}{L}x, & 0 \le x \le \frac{L}{2} \\ 2h - \frac{2h}{L}x, & \frac{L}{2} < x \le 1 \end{cases} \\ (3) & u_t(x,0) = 0, \quad 0 \le x \le 1 \\ (4) & u(0,t) = u(L,t) = 0,\ t > 0 \end{cases}$$

(initial profile is )

(since string is intially at rest.)

(ends held fixed)

P
L

<u>Separation of variables</u>: Let $U(x,t) = v(x)w(t)$ be a nonzero solution of (1), (3), & (4). (1) becomes

$$v(x)w''(t) - c^2 v''(x)w(t) = 0$$
$$v(x)w''(t) = c^2 v''(x)w(t)$$
$$\frac{w''(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)} \qquad (\text{when } v, w \ne 0).$$

Since this is true for all x, t, this equals a constant λ. Then

(5) $v''(x) - \lambda v(x) = 0, \quad 0 \le x \le 1$

(6) $v(0) = v(L) = 0$ since $u(0,t) = v(0)w(t) = 0$, $u(L,t) = v(L)w(t) = 0$.

If $\lambda > 0$ then $v(x) = Ae^{\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x}$, and (6) gives the linearly independent equations $A + B = 0$, $Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0$ so $A = B = 0$ and $v(x) = 0$. If $\lambda = 0$ then $v = Ax + B$, and (6) forces $A = B = 0$. So the only meaningful solution is when $\lambda < 0$. Let $\lambda = -\mu^2$. Then $v(x) = A\sin(\mu x) + B\cos(\mu x)$. Since we want $v(0) = v(L) = 0$, we take $B = 0$ and $\mu = \frac{\pi m}{L}$, $m \in \mathbb{N}$. We get the fundamental solutions

(7) $v(x) = \sin\left(\frac{\pi m}{L}x\right), \quad m \in \mathbb{N}, \quad \lambda_m = -\left(\frac{\pi m}{L}\right)^2$

Then the corresponding solution w satisfies

$$w''(t) - c^2\lambda_m w(t) = 0$$

(8) $\Rightarrow w''(t) + \frac{c^2\pi^2 m^2}{L^2}w(t) = 0$

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(5.1, cont.) so

$$w(t) = A \sin\left(\frac{c\pi m}{L} \cdot t\right) + B\cos\left(\frac{c\pi m}{L} t\right)$$

From (3), $v(x)\, w_t(0) = 0$ so $W_t(0) = 0$, giving $A \cdot \frac{c\pi m}{L} = 0$ and $w(t) = B\cos\left(\frac{c\pi m}{L} \cdot t\right)$.

Hence any function of the form

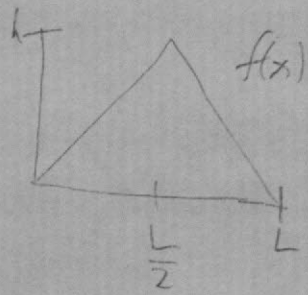
$$(9) \quad \boxed{u(x,t) = \sum_{m=1}^{\infty} \underbrace{B_m \cos\left(\frac{c\pi m}{L} \cdot t\right)}_{W_m(t)} \underbrace{\sin\left(\frac{\pi m}{L} \cdot x\right)}_{v_m(t)}}$$

solves (1), (3), and (4)

Expand $f(x) = u(x,0)$ into sine Fourier series:

$$f(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{\pi m}{L} \cdot x\right)$$

$$a_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx$$



If m is even,

$$a_m = \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx + \int_{\frac{L}{2}}^{L} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx\right)$$

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx + \int_{\frac{L}{2}}^{L} f(L-x) \cdot -\sin\left(\frac{\pi m}{L}(L-x)\right) dx\right)$$

since $\sin\left(\frac{\pi m}{L}(L-x)\right) = \sin\left(\pi m - \frac{\pi m}{L} x\right) = \sin\left(-\frac{\pi m}{L} x\right)$

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} x\right) dx + \int_{\frac{L}{2}}^{0} f(y) \cdot -\sin\left(\frac{\pi m}{L} \cdot y\right) \cdot -dy\right) = 0.$$

If m is odd,

$$a_m = \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx + \int_{\frac{L}{2}}^{L} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx\right)$$

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx + \int_{\frac{L}{2}}^{L} f(L-x) \cdot \sin\left(\frac{\pi m}{L}(L-x)\right) dx\right)$$

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx + \int_{\frac{L}{2}}^{0} f(y) \cdot \sin\left(\frac{\pi m}{L} \cdot y\right) dy\right)$$

$$= \frac{4}{L} \int_0^{\frac{L}{2}} f(x) \sin\left(\frac{\pi m}{L} \cdot x\right) dx$$

$$= \frac{4}{L} \int_0^{\frac{L}{2}} \left(\frac{2h}{L} x\right) \sin\left(\frac{\pi m}{L} \cdot x\right) dx$$

$$= \frac{8h}{L^2}\left(\left[x \cdot -\frac{L}{\pi m} \cos\left(\frac{\pi m}{L} \cdot x\right)\right]_0^{L/2} - \int_0^{\frac{L}{2}} -\frac{L}{\pi m} \cos\left(\frac{\pi m}{L} x\right) dx\right)$$

(Integration by parts, $u = x$, $dv = \sin\left(\frac{\pi m}{L} x\right) dx$)

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1 (5.1, cont.)

$$= \frac{8h}{L^2}\left(\frac{L}{\pi m}\int_0^{L/2} \cos\left(\frac{\pi m}{L}x\right)dx\right) \qquad \left(\text{noting } \cos\left(\frac{\pi m}{L}x\right)=0 \text{ for } x=\frac{L}{2}, 0 \text{ since } m \text{ odd}\right).$$

$$= \frac{8h}{L^2}\cdot\frac{L^2}{\pi^2 m^2}\left[\sin\left(\frac{\pi m}{L}x\right)\right]_0^{\frac{L}{2}}$$

$$= \frac{8h}{\pi^2 m^2}\sin\left(\frac{\pi m}{2}\cdot x\right) = \frac{8h}{\pi^2 m^2}(-1)^{\frac{m-1}{2}}$$

Hence

$$(10) \quad \boxed{f(x) = \sum_{\substack{m \text{ odd} \\ m\geq 1}} \frac{8h}{\pi^2 m^2}(-1)^{\frac{m-1}{2}}\sin\left(\frac{\pi m}{L}\cdot x\right)}$$

Use initial conditions: Take $u(x,t)$ as in (9); to satisfy (2)

we need $u(x,0)=f(x)$, or

$$f(x)=\sum_{m=1}^{\infty} B_m \sin\left(\frac{\pi m}{L}\cdot x\right) = \sum_{\substack{m \text{ odd} \\ m\geq 1}} \frac{8h}{\pi^2 m^2}(-1)^{\frac{m-1}{2}}\sin\left(\frac{\pi m}{L}\cdot x\right)$$

$$\Rightarrow \quad B_m = \begin{cases} 0, & m \text{ even} \\ \frac{8h}{\pi^2 m^2}(-1)^{\frac{m-1}{2}}, & m \text{ odd}\end{cases}$$

$$\Rightarrow u(x,t) = \sum_{\substack{m\geq 1 \\ m \text{ odd}}} \frac{8h}{\pi^2 m^2}(-1)^{\frac{m-1}{2}}\cos\left(\frac{c\pi m}{L}t\right)\sin\left(\frac{\pi m}{L}\cdot x\right)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8h}{\pi^2(2k+1)^2}(-1)^k \cos\left(\frac{c\pi(2k+1)}{L}t\right)\sin\left(\frac{c\pi(2k+1)}{L}\cdot x\right)}$$

$$E(t) = E_{kin}(t)+E_{pot}(t) = E_{kin}(0)+E_{pot}(0)$$

(energy is constant since $u_{tt}-c^2u_{xx}=0$, u constant at endpoints.)

$$= \frac{1}{2}\int_0^L \rho_0 u_t^2(x,0) + \rho_0 c^2 u_x^2(x,0)\, dx$$

$$= \frac{1}{2}\int_0^L \rho_0 c^2 u_x^2(x,0)\, dx$$

$$= \frac{1}{2}\int_0^L \rho_0 c^2\left(\frac{2h}{L}\right)^2 dx$$

$$= \frac{2h^2\rho_0 c^2}{L} \text{ or } \frac{2h^2 \tau_0}{L}$$

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5 extra credit

2. (5.2)
$$\begin{cases} (1)\; u_{tt} - u_{xx} = 0, & 0<x<1,\ t>0 \\ (2)\; u(x,0) = u_t(x,0) = 0, & 0 \le x \le 1 \\ (3)\; u_x(0,t) = 1,\ u(1,t) = 0, & t \ge 0 \end{cases}$$

u solves these equations iff $z = u - x + 1$ solves the following:

$$\begin{cases} (1')\; z_{tt} - z_{xx} = 0 & , 0<x<1, t>0 \\ (2')\; \overset{(a)}{z}(x,0) = -x+1,\ \overset{(b)}{z_t}(x,0) = 0 & , 0 \le x \le 1 \\ (3')\; z_x(0,t) = z(1,t) = 0 & , t \ge 0. \end{cases}$$

[Step 1] Separation of variables: Let $z(x,t) = v(x)w(t)$ be a nonzero solution of (1'), (2b) & (3').

(1') gives
$$v(x)w''(t) - v''(x)w(t) = 0$$
$$\Rightarrow \frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}$$
Since this is true for all x,t, this equals a constant λ. Then

$$(4)\quad v''(x) - \lambda v(x) = 0, \quad 0<x<1.$$

$$(5)\begin{cases} v_x(0) = 0 & , \text{ since } v_x(0)w(t) = 0 \text{ from } (3') \\ v(1) = 0 & , \text{ since } v(1)w(t) = 0 \text{ from } (3'). \end{cases}$$

If $\lambda = \mu^2 > 0$ then $v(x) = Ae^{\mu x} + Be^{-\mu x}$ and the boundary conditions give the linearly independent equations $\mu(A-B)=0$, $Ae^{\mu} + Be^{-\mu} = 0$, so $A = B = 0$ and $v = 0$. If $\lambda = 0$ then $v(x) = Ax + B$ and (5) gives $A=0$ and $B=0$. Hence the only nonzero solutions arise when $\lambda = -\mu^2 < 0$. Then $v(x) = A\sin(\mu x) + B\cos(\mu x)$. (5) gives $\mu A = 0 \Rightarrow A = 0$ and $B\cos(\mu) = 0$. Hence $\mu = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{N}$; giving the solution
$$v(x) = \cos\left(\frac{(2k+1)\pi}{2}x\right)$$
The corresponding solution w satisfies
$$w''(t) - \lambda w(t) = 0$$
$$w''(t) + \mu^2 w(t) = 0$$
$$\Rightarrow w(t) = A\sin\left(\frac{(2k+1)\pi}{2}t\right) + B\cos\left(\frac{(2k+1)\pi}{2}t\right)$$
From (2'), $v(x)w_t(0) = z_t(x,0) = 0$. Hence $w_t(0) = A\cdot\frac{(2k+1)\pi}{2} = 0 \Rightarrow A = 0$ and

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(5.2, cont)

$$w(t) = B\cos\left(\frac{(2k+1)\pi}{2} t\right)$$

Hence any function of the form

$$(6) \quad \boxed{z(x,t) = \sum_{k=1}^{\infty} B_k \underbrace{\cos\left(\frac{(2k+1)\pi}{2} t\right)}_{w_k(t)} \underbrace{\cos\left(\frac{(2k+1)\pi}{2} x\right)}_{v_k(t)}}$$

solves (1'), (2')(b'), and (3')

Step 2] Expand $z(x,0)$ in cosine Fourier series.

We expand $-x+1$ in cosine Fourier series on the interval $[0,2]$.

Note the average value of $-x+1$ on $[0,2]$ is 0 so there is no constant term.

$$-x+1 = \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k}{2} \cdot x\right)$$

$$a_m = \int_0^2 (-x+1)\cos\left(\frac{\pi k}{2} \cdot x\right) dx$$

$$= -\int_0^2 x\cos\left(\frac{\pi k}{2} x\right) dx \qquad \left(\text{since } \int_0^2 \cos\left(\frac{\pi k}{2} x\right) = \left[\frac{-2}{\pi k}\sin\left(\frac{\pi k}{2} x\right)\right]_0^2 = 0\right)$$

$$= -\left(\left[x \cdot \frac{2}{\pi k}\sin\left(\frac{\pi k}{2} x\right)\right]_0^2 - \int_0^2 \frac{2}{\pi k}\sin\left(\frac{\pi k}{2} x\right) dx\right) \qquad \left(\text{integration by parts}\right)$$

$$= \frac{2}{\pi k}\int_0^2 \sin\left(\frac{\pi k}{2} x\right) dx$$

$$= \frac{4}{\pi^2 k^2}\left[-\cos\left(\frac{\pi k}{2} x\right)\right]_0^2$$

$$= \frac{4}{\pi^2 k^2}\left(1 - \cos(\pi k)\right) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{8}{\pi^2 k^2} & \text{if } k \text{ is odd.} \end{cases}$$

$$\Rightarrow (7) \quad \boxed{-x+1 = \sum_{k \text{ odd}} \frac{8}{\pi^2 k^2}\cos\left(\frac{\pi k}{2} \cdot x\right) = \sum_{k=1}^{\infty} \frac{8}{\pi^2 (2k+1)^2}\cos\left(\frac{(2k+1)\pi}{2} x\right)} \quad \text{for } x \in [0,2]$$

Step 3] Use initial conditions: By $(2')^{(a)}$, we need to equate (6) and (7) when $t=0$:

$$\sum_{k=1}^{\infty} B_k \cos\left(\frac{(2k+1)\pi}{2} x\right) = z(x,0) = \sum_{k=1}^{\infty} \frac{8}{\pi^2 (2k+1)^2}\cos\left(\frac{(2k+1)\pi}{2} x\right)$$

$$\rightarrow B_k = \frac{8}{\pi^2 (2k+1)^2}$$

$$\Rightarrow \boxed{z(x,t) = \sum_{k=1}^{\infty} \frac{8}{\pi^2 (2k+1)^2}\cos\left(\frac{(2k+1)\pi}{2} t\right)\cos\left(\frac{(2k+1)\pi}{2} x\right).}$$

(5.2, cont.)

[Step 4] Finish:

$$u(x,t) = z(x,t) + x - 1$$

$$= (x-1) + \sum_{k=0}^{\infty} \frac{8}{\pi^2 (2k+1)^2} \cos\left(\frac{(2k+1)\pi}{2} t\right) \cos\left(\frac{(2k+1)\pi}{2} x\right).$$

(Notice that the initial condition for u_x is satisfied in the L^2 sense)

3 Assume $u(x,t) = v(x)w(t)$, then

$$\begin{cases} vw'' - v''w = g(t)\sin x & ① \\ w(0) = w'(0) = 0 & ② \\ v(0) = v(\pi) = 0 & ③ \end{cases}$$

①③ $\Rightarrow v = a\sin x$

$$\Rightarrow \begin{cases} w'' + w = g(t)/a & ④ \\ w(0) = w'(0) = 0 & ⑤ \end{cases}$$

Two solutions to $w''+w=0$ are $\sin t$ and $\cos t$.

Let $w = w_1\sin t + w_2\cos t$, then ④⑤ are equivalent to

$$\begin{cases} (w_1'\sin t + w_2'\cos t)' + w_1'\cos t - w_2'\sin t = g(t)/a \\ w_2(0) = 0, \quad w_1(0) + w_2'(0) = 0 \end{cases}$$

We let

$$\begin{cases} w_1'\sin t + w_2'\cos t = 0 \\ w_1'\cos t - w_2'\sin t = g(t)/a \end{cases}$$

Then

$$\begin{cases} w_1' = \frac{1}{a}g(t)\cos t \\ w_2' = -\frac{1}{a}g(t)\sin t \end{cases} \Rightarrow w_2'(0) = 0 \Rightarrow w_1(0) = 0$$

Using $w_1(0) = w_2(0) = 0$, we get $\begin{cases} w_1 = \frac{1}{a}\int_0^t g(\tau)\cos\tau\, d\tau \\ w_2 = -\frac{1}{a}\int_0^t g(\tau)\sin\tau\, d\tau \end{cases}$

Thus $w(t) = \frac{1}{a}\int_0^t \left(g(\tau)\cos\tau\sin t - g(\tau)\sin\tau\cos t\right) d\tau = \frac{1}{a}\int_0^t g(t-\tau)\, d\tau$

Hence $u(x,t) = \sin x \int_0^t g(t-\tau)\sin\tau\, d\tau$

4. (5.4) Let $u = u(x,t)$ be the solution of the global Cauchy problem for the equation $u_{tt} - c^2 u_{xx} = 0$, with initial data $u(x,0) = g(x), u_t(x,0) = h(x)$. Assume that g and h are smooth functions with compact support contained in the interval (a,b). Show that there exists T such that for $t \geq T$,

$$E_{cin}(t) = E_{pot}(t).$$

Proof. From the d'Alembert formula, we know that

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

We can write the potential and kinetic energies as

$$E_{pot} = \frac{1}{2}\int_{\mathbb{R}} \tau_0 u_x^2 dx \qquad\qquad E_{cin} = \frac{1}{2}\int_{\mathbb{R}} \rho_0 u_t^2 dx$$

where $c^2 = \frac{\tau_0}{\rho_0}$. Without loss of generality, we may set $\rho_0 = 1$ so $\tau_0 = c^2$.
We can write the d'Alembert formula as $u(x,t) = F(x+ct) + G(x-ct)$ where

$$F(z) = \frac{1}{2}g(z) + \frac{1}{2c}\int_0^z h(y)dy \qquad\qquad G(z) = \frac{1}{2}g(z) + \frac{1}{2c}\int_z^0 h(y)dy$$

So we can write the kinetic and potential energies as

$$\begin{aligned} E_{pot} &= \frac{c^2}{2}\int_R \left(F' + G'\right)^2 dx \\ E_{cin} &= \frac{c^2}{2}\int_R \left(F' - G'\right)^2 dx \end{aligned}$$

So $E_{pot}(t) = E_{cin}(t)$ if and only if

$$\int_R F'(x+ct)G'(x-ct)dx = 0$$

Now, we know that $Supp(F')$ and $Supp(G')$ are contained in $Supp(g) \cup Supp(h) \subseteq (a,b)$. Moreover, for each $z \notin (a,b)$, $F'(z) = G'(z) = 0$. So consider $t > \frac{b-a}{2c}$. Then for any $x \in (a,b)$, exactly one of $x+ct$ and $x-ct$ is outside the interval (a,b). Thus, for $t > T = \frac{b-a}{2c}$, we know that $F'(x+ct)G'(x-ct) = 0$. So for $t > T$, we have

$$\int_{\mathbb{R}} F'(x+ct)G'(x-ct)dx = 0$$

and we conclude that $E_{cin} = E_{pot}$. □