18.152: Fall 2010
Homework 6

Available	Tuesday, November 2	Due	Tuesday, November 9

Turn in the homework at the beginning of class on Tuesday, November 9. No late homework is accepted unless previously arranged with the instructor.
This week homework will cover old material and new material from page 132–141.

1. Problem 3.16 in textbook.

2. Problem 3.18 in textbook

3. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain containing the origin and let $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}$ the corresponding external domain. Prove that the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_e \\ u = g & \text{on } \partial\Omega_e \\ u \quad \text{bounded in } \Omega_e \end{cases} \tag{0.1}$$

with g continuous, has at most one solution in $C^2(\Omega_e) \cap C(\overline{\Omega}_e)$, by following the steps below:

(a) Solve the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } C_{r,R} \\ v = 0 & \text{on } \partial B_r \\ v = 1 & \text{on } \partial B_R, \end{cases} \tag{0.2}$$

in the annulus $C_{r,R} = B_R \setminus \overline{B_r}$, where $B_r \subset \Omega \subset B_R$, and the disks B_r and B_R are centered at the origin.
Hint: You will find that the solution is radially symmetric and in particular

$$v(\rho) = \frac{\ln(\rho/r)}{\ln(R/r)}.$$

(b) Prove that , if u_1 and u_2 are two solutions to (0.1) and

$$w = \begin{cases} u_1 - u_2 & \text{in } \Omega_e \\ 0 & \text{in } \Omega, \end{cases}$$

then

$$|w| \le Mv \quad \text{in } C_{r,R} \tag{0.3}$$

where v is the solution to (0.2) and M is an appropriate constant.

(c) Now let R go to infinity in (0.3) to deduce that $w = 0$ and hence (0.1) has a unique solution.

Turn page, there is one more problem!

4. Find a formula for the Green's function for the following domains:

(a) The half plane $P_a = \{(x_1, x_2) \,:\, x_1 > a\}$.

(b) The disk B_P of radius R and center P in the plane.

(c) The half disk B_1^+ defined in Problem 3 above.