40/40 Congratulations! Excellent Work!

18.152 PSET 6 DUE 11/9 [redacted]

20/20

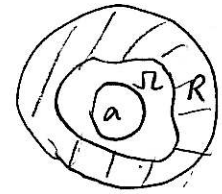
1. 1) Let u_1, u_2 be two solutions. $w = u_1 - u_2$. Then $\begin{cases} \Delta w = 0 \text{ in } \Omega_e \\ w = 0 \text{ on } \partial\Omega \\ w \text{ bounded in } \Omega_e \end{cases}$ ✓

Say $|w| \le M$ in Ω_e.

Clearly, $u_R(x)$ is harmonic, i.e. $\Delta u_R = 0$ in Ω_e. Let $v = u_R - w$

Thus $\Delta v = 0$ in Ω_e. Consider the bounded domain $\Omega' = B_{a,R} \cap \Omega_e$. $v \in C^2(\Omega') \cap C(\overline{\Omega'})$

In Ω', $|w| \le M$, $0 \le u_R \le M$, thus $|v| \le 2M$. ✓



$\partial\Omega' = \partial\Omega_e \cup \partial B_R(0)$

On $\partial\Omega_e$, $w = 0 \Rightarrow v = u_R - w = u_R \ge 0$

On $\partial B_R(0)$, $u_R = M \Rightarrow v = u_R - w = M - w \ge 0$.

Hence $\begin{cases} \Delta v = 0 \\ v \ge 0 \text{ on } \partial\Omega' \\ v \text{ bounded in } \Omega' \end{cases}$

By the maximal principle, $v \ge 0$ in Ω', i.e. $w \le u_R$ in $B_{a,R} \cap$.

2) Fix $x \in \Omega_e$. For any $R > |x|$, $w(x) \le u_R(x) \to 0$ as $R \to \infty \Rightarrow w(x) \le$ ✓

3) Let $\bar{u}_R(x) = -u_R(x)$. Then $-M \le \bar{u}_R \le 0$. Let $\bar{v} = w - \bar{u}_R$ ✓

We still have $\begin{cases} \Delta \bar{v} = 0 \\ \bar{v} \ge 0 \text{ on } \partial\Omega' \\ \bar{v} \text{ bounded in } \Omega' \end{cases}$

Thus $\bar{v} \ge 0$ in Ω', i.e. $w \ge \bar{u}_R$ in $B_{a,R} \cap \Omega_e$.

Fix $x \in \Omega_e$. For any $R > |x|$, $w(x) \ge \bar{u}_R(x) \to 0$ as $R \to \infty \Rightarrow w(x) \ge$

✓

3.18 Suppose $0 \in \Omega$.

for R large enough, and $\Omega \subset\subset B_R(0)$

a). By Green's identity,

$$\int_{\partial B_R(0) \cup \partial\Omega} \partial_\nu u(\sigma)\, d\sigma = \int_{\Omega_e \cap B_R(0)} \Delta u = 0$$

$$\therefore \int_{\partial B_R(0)} \partial_\nu u(\sigma)\, d\sigma + \int_{\partial\Omega} g = 0$$

From corr (3.2), we have

$$|u_{x_i}(x)| \leq \frac{c}{|x|} \max_{x \in B_R(0)} u(x)$$

since we have

$$|u(x)| \leq \frac{1}{|R|^{1+\varepsilon}}$$

$\therefore$ we have

$$|\partial_\nu u(x)| \leq \frac{3c}{|R|} \cdot \frac{1}{|R|^{1+\varepsilon}} = \frac{3c}{|R|^{2+\varepsilon}}$$

$$\therefore \int_{\partial B_R(0)} \partial_\nu u(\sigma)\, d\sigma \leq 4\pi R^2 \cdot \frac{3c}{|R|^{2+\varepsilon}}$$

$$= \frac{12\pi c}{|R|^{\varepsilon}}$$

$\therefore$ when $R \to +\infty$,

$$\int_{\partial\Omega} g = -\int_{\partial B_R(0)} \partial_\nu u(\sigma)\, d\sigma \to 0$$

$$\therefore \int_{\partial\Omega} g\, d\sigma = 0.$$

b). Since we write it in a single layer potential,

$$\therefore u(x) = \int_{\partial\Omega} \phi(x-\sigma)\, v(\sigma)\, d\sigma$$

since

$$g(x) = \partial_\nu u + ku \quad \text{on } \partial\Omega_e = \partial\Omega$$

by P. 147

$$\partial_\nu u(x) = -\frac{v(x)}{2} + \int_{\partial\Omega} \frac{\partial}{\partial n_x} \phi(x-\sigma)\, v(\sigma)\, d\sigma + k\int_{\partial\Omega} \phi(x-\sigma)\, v(\sigma)\, d\sigma$$

this is the integral function of the density v.

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18.152 PSet 6

2020

(Green's functions)

Let

$$\Phi(x) = \frac{-1}{2\pi} \ln|x|$$

be the fundamental solution with $\Delta u = \delta_0$ in $\mathbb{R}^2$. We look for

$$G(x,y) = \Phi(x-y) - \varphi(x,y)$$

so that $\Delta_y G(x,y) = -\delta_x$ in the given domain Ω and $G(x,y)=0$ for $y \in \partial\Omega$; equivalently, $\Delta_y \varphi = 0$ in Ω and $\varphi(x,y) = \Phi(x-y)$ for $y \in \partial\Omega$. ✓

(a) $P_a = \{(x_1, x_2) : x_1 > a\}$

6/6 For a point $x = (x_1, x_2)$ let

$$x^* = (2a - x_1, x_2)$$

be its reflection across $x_1 = a$. Note that if $x \in P_a$ then $x^* \notin P_a$.

Let

$$\varphi(x,y) = \frac{-1}{2\pi} \ln|x^* - y|.$$

Then

$$\Delta_y \varphi(x,y) = -\delta_{x^*}(y) = 0 \quad \text{for } x,y \in P_a \text{ (since } x^* \notin P_a)$$

and for $y = (y_1, y_2) \in \partial P_a$, $x = (x_1, x_2) \in P_a$.

$$\varphi(x,y) = \frac{-1}{2\pi} \ln|(2a-x_1, x_2) - (a, y_2)| \qquad |x_1 - a| = |(2a-x_1) - a|.$$

$$= \frac{-1}{2\pi} \ln|(x_1, x_2) - (a, y_2)|$$

$$= \Phi(x-y) \quad ✓$$

The Green's function is

$$\boxed{G(x,y) = \frac{1}{2\pi}\left(-\ln|x-y| + \ln|x^* - y|\right) \quad \left(\begin{array}{l} x = (x_1, x_2) \\ x^* = (2a - x_1, x_2) \end{array}\right)}$$

✓

7/7 (b) $B_p = \{x \in \mathbb{R}^2 \mid |x - p| < R\}$.

We look for $\varphi(x,y)$ in the form

$$\varphi(x,y) = \frac{-1}{2\pi}\left(\ln|x^+ - y| + k\right) \quad \text{(where } x^+, k \text{ depend only on } x)$$

$$= \frac{-1}{2\pi} \ln(C|x^+ - y|) \quad \text{where } C = e^k > 0$$

When $y \in \partial B_p$, we need

$$\varphi(x,y) = \Phi(x-y)$$

$$\Leftrightarrow \ln(C|x^+ - y|) = \ln|x - y|$$

$$\Leftrightarrow C|x^+ - y| = |x - y| \quad ✓$$

[redacted]

(b, cont) $\iff C^2|x^+-y|^2 = |x-y|^2$

$\iff C^2|(x^+-p)-\sigma|^2 = |(x-p)-\sigma|^2$ where $\sigma = y-p$.

$$\iff C^2\left(|x^+-p|^2-2(x^+-p)\cdot\sigma+\underbrace{\cancel{|\sigma|^2}}_{R^2}\right) = |x-p|^2-2(x-p)\cdot\sigma+\underbrace{\cancel{|\sigma|^2}}_{R^2}$$

$$(*) \qquad \iff |x-p|^2 - C^2\left(|x^+-p|^2+R^2\right) + R^2 = 2\sigma\cdot\left((x-p)-C^2(x^+-p)\right)$$

In order for this to be true for all $y \in \partial B_p$, i.e. all σ such that $|\sigma| = R$, we must have $(x-p) - C^2(x^+-p) = 0$:

$$x-p = C^2(x^+-p).$$

Putting this in the LHS of $(*)$ and setting equal to 0,

$$|x-p|^2 - \frac{1}{C^2}|x-p|^2 + R^2(1-C^2) = 0.$$

$$\iff \left(\frac{C^2-1}{C^2}\right)|x-p|^2 = R^2(C^2-1)$$

$$\iff C = \frac{|x-p|}{R}.$$

Thus we set $x^+ - p = \frac{R^2}{|x-p|^2}(x-p)$, i.e. $x^+ = \frac{R^2}{|x-p|^2}(x-p)$. Note $x \in B_p$ implies that $|x-p| < R$ and $|x^+-p| = \frac{R^2}{|x-p|} > R$, giving $x^+ \notin B_p$.

Hence for $y \in B_p$, and C depending on x,

$$\Delta_y \varphi(x,y) = -C\,\delta_{x^+}(y) = 0,$$

$$\boxed{G(x,y) = \frac{1}{2\pi}\left(-\ln|x-y| + \ln\left(\frac{|x-p|}{R}|x^+-y|\right)\right), \quad x^+ = \frac{R^2}{|x-p|^2}(x-p)+p}$$

This is defined for $x \neq p$. To define $G(p,y)$, take the limit:

$$\lim_{x\to p} G(x,y) = \frac{-1}{2\pi}\left(\ln|p-y| + \lim_{x\to p}\left(\ln\left[\frac{|x-p|}{R}\left|\frac{R^2}{|x-p|^2}(x-p)+p-y\right|\right]\right)\right)$$

$$= \frac{-1}{2\pi}\left(\ln|p-y| + \lim_{x\to p}\left(\ln\left|\frac{R}{|x-p|}(x-p) + \underbrace{\frac{|x-p|}{R}}_{\to 0}(p-y)\right|\right)\right)$$

$$= \frac{-1}{2\pi}\left(\ln|p-y| + \ln R\right)$$

Define $G(p,y) = \frac{-1}{2\pi}\left(\ln|p-y| + \ln R\right)$

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7/7

(c) $B_1^+ = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1, x_2 > 0\}$

By (a) with x_1, x_2 switched and $a = 0$, the Green's function for the upper half plane S is

$$G_1(x,y) = \frac{1}{2\pi}\left(-\ln|x-y| + \ln|x^*-y|\right), \quad x = (x_1, x_2), \; x^* = (x_1, -x_2).$$

The Green's function for the disc B_1 of radius 1 centered at 0 is, by (b),

$$G_2(x,y) = \frac{1}{2\pi}\left(-\ln|x-y| + \ln|x|\,|x^+ - y|\right), \quad x^+ = \frac{1}{|x|^2}x. \checkmark$$

We show that

$$\boxed{\begin{array}{l} G(x,y) = \dfrac{1}{2\pi}\left(-\ln|x-y| + \ln|x^*-y| + \overbrace{\ln|x^+-y| - \ln|x^{*+}-y|}^{\text{note this equals } \ln(|x|\,|x^+-y|) - \ln(|x^*|\,|x^{*+}-y|) \text{ since } |x| = |x^*|}\right) \\ x = (x_1, x_2) \qquad x^+ = x/|x|^2 \\ x^* = (x_1, -x_2) \qquad x^{*+} = (x_1, -x_2)/|x|^2 \end{array}}$$

$\checkmark$

is a Green's function for B_1^+:

- $\Delta_y G(x,y) = -\delta_x$ in B_1^+: This follows since $\Delta_y G(x,y) = -\delta_x + \delta_{x^*} + \underbrace{C_1 \delta_{x^+} - C_2 \delta_{x^{*+}}}_{\text{constants depending on } x \text{ only.}}$ and $x^*, x^+, x^{*+} \notin B_1^+$. $\checkmark$
- $G(x,y) = 0$ for $x \in B_1^+$, $y \in \partial B_1^+$
  - When y is on the circumference, $y \in \partial B_1$, then

$$G(x,y) = G_2(x,y) - G_2(x^*,y) = 0 - 0 = 0$$

since G_2 is a Green's function for B_1, and $x, x^* \in B_1$. $\checkmark$

  - When $y = (y_1, 0) \in \partial B_1^+$, then

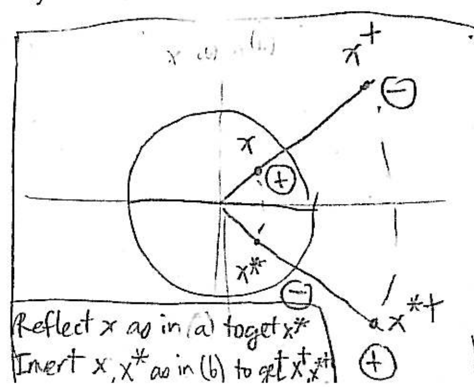
$$G(x,y) = G_1(x,y) - G_1(x^+,y) = 0 - 0 = 0$$

$\checkmark$ since G_1 is Green's function for S, and $x, x^+ \in S$.

At $(0, y)$, we set

$$G(0,y) = G_2(0,y) - G_2(0^*,y) = 0$$

as in the end of (b). $\checkmark$



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