1. (3.6) Let

$$B_1^+ = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1, y > 0\}$$

and $u \in C^2(B_1^+) \cap C(\overline{B_1^+})$, harmonic in B_1^+, $u(x,0) = 0$. Show that the function

$$U(x,y) = \begin{cases} u(x,y) & y \geq 0 \\ -u(x,-y) & y < 0 \end{cases}$$

obtained by u by odd reflection with respect to y, is harmonic in all of B_1.

Proof. Let v be the solution to

$$\begin{cases} \Delta v = 0 & \text{in } B_1 \\ v = U & \text{on } \partial B_1. \end{cases}$$

Now let $w(x,y) = v(x,y) + v(x,-y)$. Notice that $\Delta w = 0$ in B_1. We claim that $w = 0$ on ∂B_1. To this end, let $(x,y) \in \partial B_1$. First suppose that $y \geq 0$. From definition of U, we know that $u(x,-y) = -u(x,y)$. So $w(x,y) = v(x,y) + v(x,-y) = U(x,y) + U(x,-y) = 0$. Suppose, now that $y < 0$. By similarly reasoning we can see that $w(x,y) = 0$, in this case.

Therefore, on ∂B_1, $w = 0$. So w is a solution to the problem

$$\begin{cases} \Delta w = 0 & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1 \end{cases}$$

Therefore, by uniqueness, we see that $w = 0$. This means that $v(x,y) = -v(x,-y)$. In particular, we see that $v(x,0) = -v(x,0)$, and conclude that $v(x,0) = 0 = U(x,0)$.

Now consider $\partial B_1^+ = \partial_1 + \partial_2$, where $\partial_1 = \{(x,y) \in \partial B_1 | y > 0\}$ and $\partial_2 = \{(x,y) \in \partial B_1 | y = 0\}$. Then on ∂_1, we know that $v = U$, and on ∂_2, we know that $v = 0 = U$. This means that v is a solution to

$$\begin{cases} \Delta v = 0 & \text{in } B_1^+ \\ v = U & \text{on } \partial B_1^+. \end{cases}$$

By uniqueness, we conclude that $v = U$ on $\overline{B_1^+}$.

By similarly reasoning with the set B_1^- (the set of points in B_1 below the x-axis), we see that $v = U$ on $\overline{B_1^{-1}}$. So $v = U$ on $\overline{B_1}$, and we conclude that U is harmonic in all of B_1. ☐

Notice that this argument can be extended to balls of any radius.

Problem 2 *(3.8, Harmonic and L^2 on $\mathbb{R}^n$)*

By the Mean Value Property, for any $R > 0$,

$$u(\mathbf{x}) = \frac{1}{\mathrm{vol}(B_R(\mathbf{x}))} \int_{B_R(\mathbf{x})} u(\mathbf{y})\, d\mathbf{y}$$

Hence

$$\begin{aligned}
\mathrm{vol}(B_R(\mathbf{x})) \int_{B_R(\mathbf{x})} u(\mathbf{y})^2\, d\mathbf{y} &= \left(\int_{B_R(\mathbf{x})} 1\, d\mathbf{y}\right)\left(\int_{B_R(\mathbf{x})} u(\mathbf{y})^2\, d\mathbf{y}\right) \\
&\geq \left(\int_{B_R(\mathbf{x})} u(\mathbf{y})\, d\mathbf{y}\right)^2 && \text{Cauchy-Schwarz inequality} \\
&= (\mathrm{vol}(B_R(\mathbf{x}))u(\mathbf{x}))^2 && \text{Mean Value Property}
\end{aligned}$$

This gives

$$\int_{B_R(\mathbf{x})} u(\mathbf{y})^2 \, d\mathbf{y} \geq \mathrm{vol}(B_R(\mathbf{x}))u(\mathbf{x})^2.$$

If $u(\mathbf{x}) \neq 0$, then letting $R \to \infty$ above we get that the right-hand side diverges, and

$$\int_{\mathbb{R}^3} u(\mathbf{y})^2 \, d\mathbf{y} = \infty.$$

Hence, if

$$\int_{\mathbb{R}^3} u(\mathbf{y})^2 \, d\mathbf{y} < \infty$$

then $u \equiv 0$. 5/5 Very Nice!

3 $u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\left|\frac{x-y}{x}\right| + \log|x|\right) f(y)\,dy$

$= -\frac{1}{2\pi}\log|x| \int_{\mathbb{R}^2} f(y)\,dy - \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left|\frac{x-y}{x}\right| f(y)\,dy$

$= -\frac{M}{2\pi}\log|x| - \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left|\frac{x-y}{x}\right| f(y)\,dy$ (*)

Note that for sufficiently large $|x|$, i.e. $|x| > |y'|$, $\forall y' \in K$

$\log\left|\frac{x-y}{x}\right| \geq \log\frac{|x|-|y|}{|x|} = \log\left(1-\left|\frac{y}{x}\right|\right) = -\left|\frac{y}{x}\right| + O\left(\left|\frac{y}{x}\right|^2\right)$

$\log\left|\frac{x-y}{x}\right| \leq \log\frac{|x|+|y|}{|x|} = \log\left(1+\left|\frac{y}{x}\right|\right) = \left|\frac{y}{x}\right| + O\left(\left|\frac{y}{x}\right|^2\right)$

Hence $\left|\log\left|\frac{x-y}{x}\right|\right| \le \left|\frac{y}{x}\right| + O\left(\left|\frac{y}{x}\right|^2\right) \le \frac{\sup_K |y|}{|x|} + O\left(\left(\frac{\sup_K |y|}{|x|}\right)^2\right)$

Thus $\exists$ $C>0$ s.t. $\log\left|\frac{x-y}{x}\right| \le \frac{C}{|x|}$, $\forall y$, for $|x| > \max\{1, \sup_K |y|\}$.

Therefore, by $(*)$, we get

$$\left|u(x) + \frac{M}{2\pi}\log|x|\right| = \left|\frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left|\frac{x-y}{x}\right| f(y)\,dy\right|$$

$$\le \frac{1}{2\pi}\left|\int_{\mathbb{R}^2} \frac{C}{|x|} f(y)\,dy\right|$$

$$= \frac{C}{2\pi|x|} M$$

$\Rightarrow \left|u + \frac{M}{2\pi}\log|x|\right| = O(|x|^{-1})$

Hence

$$u = -\frac{M}{2\pi}\log|x| + O(|x|^{-1})$$

4(a) $\frac{5}{5}$ If there exist two solutions u_1 and u_2. Let $u = u_1 - u_2$.

Then $\begin{cases} \Delta u = \Delta u_1 - \Delta u_2 = 0, x \in \mathbb{R}, 0 < y \\ u(x,0) = u_1(x,0) - u_2(x,0) = g(x) - g(x) = 0, \quad x \in \mathbb{R}. \\ u(x,y) = u_1(x,y) - u_2(x,y) \text{ is bounded in } S \end{cases}$

Let $\bar{u}(x,y) = \begin{cases} u(x,y) & , y > 0 \\ -u(x,-y) & , y < 0 \\ 0 & , y = 0 \end{cases}$

Then $\bar{u} \in C^2(B_r^+) \cap C(\overline{B_r^+})$, harmonic in B_r^+, $u(x,0)$ for all $r > 0$

Applying the first Ex. with 1 replaced by r, we see that $\bar{u}$ is harmonic in all B_r's. Let $r \to +\infty \Rightarrow \bar{u}$ is harmonic in $\mathbb{R}^2$

Since $\bar{u}$ is bounded, $\exists M \in \mathbb{R}$ s.t. $\bar{u} \geq M$ on $\mathbb{R}^2$. By Liouville's Thm, $\bar{u}$ is a constant $\Rightarrow \bar{u} = \bar{u}(x,0) = 0 \Rightarrow u_1 = u_2$

This completes the proof.

~~Thus~~ G is continuous on $[\frac{\pi}{2}, \frac{5\pi}{2}]$, so continuous on $\partial B_1(0)$ when $\begin{cases} x=\cos\theta \\ y=\sin\theta \end{cases}$

Then V is a solution to $\begin{cases} \Delta V = 0 \\ V(\cos\theta, \sin\theta) = G(\theta) \quad \text{on } \partial \bar{B}_1(0) \end{cases}$, $\begin{cases} x=\cos\theta \\ y=\sin\theta \end{cases}$.

This problem has a unique solution.

Therefore $u(x,y) = V(f(x,y))$ has a unique solution.

(b) $$\hat{u}_{yy}(\xi, y) = \int_{\mathbb{R}} e^{-i\xi x} u_{yy}(x,y)\,dx$$

$$= -\int_{\mathbb{R}} e^{-i\xi x} u_{xx}(x,y)\,dx$$

$$= -\int_{\mathbb{R}} e^{-i\xi x}\, d u_x = \int_{\mathbb{R}} u_x\, e^{-i\xi x}(-i\xi)\,dx$$

$$= -i\xi \int_{\mathbb{R}} e^{-i\xi x}\, du = -(-i\xi)^2 \int_{\mathbb{R}} e^{-i\xi x} u\,dx = \xi^2 \int_{\mathbb{R}} e^{-i\xi x} u\,dx$$

$$= \xi^2 \hat{u}(\xi, y).$$

$$\Rightarrow \hat{u}_{yy} - \xi^2 \hat{u} = 0$$

Let $\hat{u} = C(\xi)V(y)$. Then $V''(y)C(\xi) - \xi^2 V(y)C(\xi) = 0 \Rightarrow V''(y) = \xi^2 V(y)$

$$\Rightarrow V(y) = e^{\pm|\xi| y}$$

$$\Rightarrow \hat{u} = C_1(\xi)e^{|\xi| y} + C_2(\xi)e^{-|\xi| y}$$

$$\overset{\text{bounded}}{\Rightarrow} \quad C_1(\xi) = 0 \Rightarrow \hat{u} = C(\xi)e^{-|\xi| y}$$

$$\overset{y=0}{\Rightarrow} \quad C(\xi) = \int_{\mathbb{R}} e^{-i\xi x} g(x)\,dx = \text{Fourier transform of } g.$$

If $e^{-|\xi| y}$ is the Fourier transform of some function $P(x,y)$, then

$$u(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)\, P(x-t, y)\,dt.$$

In fact, we find $P(x,y) = \frac{2y}{x^2+y^2}$ (the inverse Fourier transform of $e^{-|\xi| y}$).

Thus $$u(x,y) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{g(t)}{(x-t)^2+y^2}\,dt.$$

5. WOLG, we can assume $X_0 = \vec{0}$.

We prove that for all $x \in \Omega$,

$\exists$ a constant C depending on x,

s.t. $u_i(x) \le C u_i(0)$

(1) Since Ω is open, $\exists R$ s.t. $B_R(0) \subset \Omega$.

For any $x \in B_R(0)$,

By Harnack's inequality,

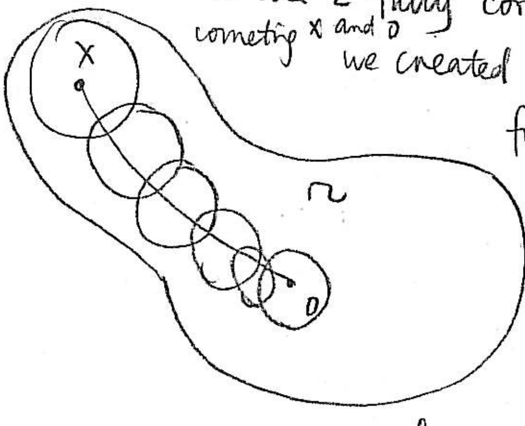
we have

$$\frac{R-|x|}{R+|x|} u_i(0) \le u_i(x) \le \frac{R+|x|}{R-|x|} u_i(0)$$

Let $C = \frac{R+|x|}{R-|x|}$, we are good.

(2) Now we show we can find such C for all $x \in \Omega$.

Since Ω is connected, we can find a line L fully contained in Ω connecting x and 0



we created a open cover for L:

$\{B_{R_s}(s) : \forall s \in L, R_s(s) \subset \Omega\}$

since L is compact, $\exists$ a finite number of neiborhoods $B_1, B_2, \cdots, B_n$

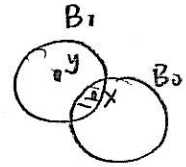
s.t. $B_i \cap B_{i+1} \neq \phi$, $\bigcup B_i \subset \Omega$, $\bigcup B_i \supset L$.

Let B_0 be the ball centered at 0 as in (1).

Then all $x \in B_0$ satisifies

$u_i(x) \le C u_i(0)$ for some C depending on x.

clearly, $B_0 \cap B_1 \neq \phi$



If B_1 is centered at y and $x \in B_0 \cap B_1$, $\exists C$ s.t. $u_i(x) \le C_1 u_i(0)$

By Harnack's inequality,

$$\frac{R-|x-y|}{R+|x+y|} u(y) \le u(x) \le \frac{R+|x-y|}{R-|x-y|} u(y)$$

$$\text{So} \quad u_i(y) \le \frac{R+|x+y|}{R-|x-y|} u(x) \le C_1 \frac{R+|x+y|}{R-|x-y|} u_i(0)$$

choose $C_1 \frac{R+|x+y|}{R-|x-y|} = C_2$

then $u_i(y) \le C_2 u_i(0)$

Then repeat the proof in (1),

we have $\exists z \in B_1$,

$$u_i(z) \le C_3 u_i(y) \le C_3 \cdot C_2 u_i(0)$$

By induction, since there are only finitely many balls $B_1, \cdots, B_n$,

we have $\exists C$ s.t. $u_i(x) \le C u_i$ for $x \in B_n$

Since x is arbitnay,

we proved all $x \in \Omega$, $\exists C$,

s.t. $u_i(x) \le C u_i(0)$

Since $\sum_{i=1}^{N} u_i(0)$ converges as $N \to +\infty$.

$u_i(x) \le C u_i(0)$

by comparison theorem,

$\sum_{i=1}^{N} u_i(x)$ converges as $N \to +\infty$

(3) For any compact $K \subset \Omega$,

Let $C_0 = \sup\{C_x : x \in K\}$,

it exists and finite since K is compact.

$\therefore$ $u_i(x) \le C_0 u_i(0)$ for all $x \in K$.

since $\sum u_i(0)$ converges to M

for any $\frac{\varepsilon}{C_0} > 0$, $\exists N$ s.t.

$\sum_{1}^{n} u_i(0) - \sum^{m} u_i(0) < \frac{\varepsilon}{C_0}$ for all $n > m > N$.

$\therefore$ $\sum_{1}^{n} u_i(x) - \sum_{1}^{m} u_i(x)$

$\le C_0 \cdot \frac{\varepsilon}{C_0} = \varepsilon$

$\therefore$ $u_i(x)$ converges uniformly in K

(4) Show the sum of the series is a nonnegative harmonic function.

Let $V_n = \sum_{i=1}^{n} u_i$, clearly $V_n \ge 0$

$\therefore$ $\lim_{n \to +\infty} \sum_{i=1}^{n} u_i(x) \ge 0$.

For any $K \subset \Omega$,

V_n converges uniformly to V.

$\therefore$ for any $\varepsilon > 0$,

$\exists N$. s.t.

$V(x) - \sum_{i=1}^{n} V_i(x) < \varepsilon$ for all $n > N$.

$\therefore$ $\lim_{n \to +\infty} \frac{\int_{B_r(x)} V_n(y)\,dy}{\pi R^2} = \frac{\int_{B_r(x)} V(y)\,dy}{\pi R^2}$

$\lim_{n \to +\infty} V_n(x) = V(x)$

since $\frac{\int V_n(y)\,dy}{\pi R^2} = V_n(x)$

we have $V(x) = \frac{\int_{B_r(x)} V(y)\,dy}{\pi R^2}$

$\therefore$ V satisfies mean value theorem,

$\therefore$ V is harmonic

$\therefore$ V is nonnegative harmonic