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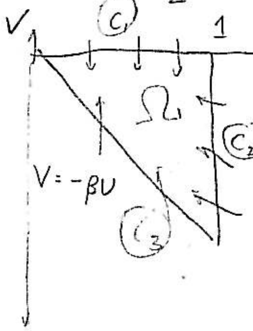
46/46 Congratulations! Excellent Work!

18.152 PSet 4

1. (2.23) $\vec{F} = V\hat{i} + (-cV + U^2 - U)\hat{j}$ 6/6

i) Let $\beta = \frac{c}{2}$ Let $C_1 = [0,1] \times \{0\}$, $C_2 = \{1\} \times (0, -c]$, and $C_3 = \{(U, -\beta U) \mid U \in (0,1)\}$



- On C_1, $\vec{F} = (U^2 - U)\hat{j}$, $\hat{n} = -\hat{j}$
  $\vec{F} \cdot \hat{n} = (U^2 - U)\hat{j} \cdot -\hat{j} = U - U^2 = U(1-U) \geq 0$
  with equality (& $\vec{F} = 0$) only at $(0,0)$ & $(1,0)$
- On C_2, $\vec{F} = V\hat{i} - cV\hat{j}$, $\hat{n} = -\hat{i}$
  $\vec{F} \cdot \hat{n} = (V\hat{i} - cV\hat{j}) \cdot -\hat{i} = -V > 0$
- On C_3, $\vec{F} = -\beta U\hat{i} + (c\beta U + U^2 - U)\hat{j}$, $\hat{n} = \frac{1}{\sqrt{\beta^2+1}}(\beta\hat{i} + \hat{j})$

$$\vec{F} \cdot \hat{n} = (-\beta U\hat{i} + ((c\beta - 1)U + U^2)\hat{j}) \cdot \frac{1}{\sqrt{\beta^2+1}}(\beta\hat{i} + \hat{j})$$
$$= \frac{1}{\sqrt{\beta^2+1}}((-\beta^2 + c\beta - 1)U + U^2) = \frac{1}{\sqrt{\beta^2+1}}((-\frac{c^2}{4} + \frac{c^2}{2} - 1)U + U^2)$$
$$= \frac{1}{\sqrt{\beta^2+1}}((\frac{c^2}{4} - 1)U + U^2) > 0 \quad \text{since} \quad c \geq 2 \Rightarrow \frac{c^2}{4} - 1 \geq 0$$

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(ii) Since $\vec{F} \cdot \hat{n} \geq 0$ along $\partial\Omega$, with equality only at $(0,0)$ and $(1,0)$ (at which $\vec{F} = 0$), Ω is a positively invariant region: orbits of

$$\begin{cases} \frac{dU}{dz} = V \\ \frac{dV}{dz} = -cV - U + U^2 \end{cases}$$

cannot leave Ω. All trajectories starting in Ω are bounded (stay in Ω), so by the Poincaré-Bendixson Theorem, a solution either converges to a critical point, is periodic, or approaches a limit cycle. The latter two are impossible since there must be a critical point in any periodic orbit, and the only critical points $(0,0)$ and $(1,0)$ are on the boundary. Since $(1,0)$ is saddle point and $(0,0)$ is a sink*, all orbits starting in Ω must converge to the origin as $z \to \infty$ (they can't converge to $(1,0)$).

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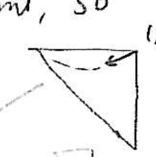
(iii) The separatrix "emanates" from $(1,0)$ in the direction of the eigenvector $(-c - \sqrt{c^2+4}, -2)$, a vector in the 3rd quadrant, so starts in Ω. Thus it approaches $(0,0)$ as $z \to \infty$.



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* The matrices of the linearizations are $\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$ with eigenvalues $\frac{1}{2}(-c \pm \sqrt{c^2+4})$; $\frac{1}{2}(-c \pm \sqrt{c^2-4})$ respectively.

2. Answer the following questions with a full explanation 16/16

(a) Let u be a solution to the equation $u_t - u_{xx} = -1$ in $0 < x < 1$, and $t > 0$ such that $u(x, 0) = 0$ and $u(0, t) = u(1, t) = \sin(\pi t)$. Is it possible that there exists a point x_0 such that $u(x_0, 1) = 1$? Suppose that there is such a point x_0. Notice that 1 is the maximum of u on the boundary. So if $u(x_0, 1) = 1$, then u achieves its maximum in $(0, 1) \times (0, \infty)$. This means that $u_t(x_0, 1) = 0$, since this is a maximum with respect to t. Similarly, from the second derivative test, we have that $u_{xx}(x_0,1) \le 0$. Therefore, at $(x_0,1)$, we have $u_t - u_{xx} \ge 0$, a contradiction to the hypothesis that $u_t - u_{xx} = -1$. Therefore, no such x_0 exists. 4/4

(b) Does the Cauchy problem

$$\begin{cases} u_t(x,t) + u_{xx}(x,t) = 0 & -1 < x < 1, 0 < t < T \\ u(x,0) = |x| & -1 < x < 1 \\ u_x(0,t) = u(1,t) = 0 & 0 < t < T \end{cases}$$

have a solution?

First observe that the operator $\partial_t - \partial_{xx}$ is smoothing (as seen in the textbook). So the operator $\partial_t + \partial_{xx}$ is smoothing but in the time reverse direction (this can be seen in discussion in textbook when considering the "backwards" heat equation). So suppose that there is a solution u to the above problem and consider the solution at some time $0 < t_0 < T$, $u(x,t_0)$. Since the operator $\partial_t + \partial_{xx}$ is smoothing in the reverse time direction. So as $t_0 \to 0$, u should remain smooth. However, the $u(x,0) = |x|$, is not smooth, so there cannot be a solution to the given problem. 4/4

(c) Check that the function $u(x,t) = \partial_x \Gamma_1(x,t)$ solves the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

and that $u(x,t) \to 0$ if $t \to 0$, for each fixed x. Is there a contradiction to the uniqueness theorem for the global Cauchy problem?

We know that $\Gamma_1(x,t)$ is a solution to $u_t - u_{xx} = 0$, and $u(x,0) = \delta$. So

$$\frac{\partial \Gamma_1(x,t)}{\partial t} - \frac{\partial^2 \Gamma_1(x,t)}{\partial x^2} = 0.$$

Differentiating with respect to x and interchanging the order of differentiation, we see that $\partial_x \Gamma_1(x,t)$ is a solution to $u_t - u_{xx} = 0$.

We know

$$\frac{\partial \Gamma_1(x,t)}{\partial x} = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} \frac{-1}{2t}$$

If we let $u = \frac{1}{t}$, then we can re-write this as

$$\frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} \frac{-1}{2t} = \frac{-u^{\frac{3}{2}}}{4\sqrt{\pi}} e^{\frac{-x^2 u}{4}}$$

Notice that this approaches 0 as $u \to \infty$, and that $u \to \infty$, as $t \to 0^+$. So $u = \partial_x \Gamma_1(x,t)$ is a solution to the given problem. However, this is not a contradiction to the uniqueness property, because u is not continuous nor is it bounded by any e^{Ax^2}. 4/4

(d) Let $(u(x,t)$ be the continuous solution of the Robin's problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & 0 < x < 1, 0 < t < T \\ u(x,0) = \sin(\pi x) & 0 \le x < 1 \\ -u_x(0,t) = u_x(1,t) = -hu, & h > 0, 0 < t < T. \end{cases}$$

Show that u cannot have a negative minimum. What is the maximum for u?

Proof. From the maximum principle, it suffices to show that the minimum of u on the boundary is non-negative. To this end, suppose that the minimum of u on the boundary occurs at (x_0, t_0) and that $u(x_0, t_0) < 0$. Then from the initial condition, we see that $t_0 > 0$ (since for $0 < x < 1$, $\sin(\pi x) \geq 0$.) So if $x_0 = 0$, then $u_x(0, t) = hu(0, t_0) < 0$. This means that for some $\varepsilon > 0$, $u(\varepsilon, t_0) < u(0, t_0)$, a contradiction since $u(x_0, t_0)$ is the minimum on the boundary (and hence, the minimum of u for the entire domain). Similarly, if $x_0 = 1$, then $u_x(1, t) = -hu(1, t_0) > 0$. This means that for some $\varepsilon > 0$, $u(1 - \varepsilon, t_0) < u(1, t_0)$, a contradiction since $u(x_0, t_0)$ is the minimum on the boundary (and, by the maximum principle, the minimum of u for the entire domain.)

Therefore, the minimum of u on the boundary must be non-negative.

For the maximum, we will show that the maximum of u on the boundary is 1. To this end, notice that 1 is the maximum value of $\sin(\pi x)$. Now suppose for the sake of contradiction that the maximum of the boundary occurs at (x_0, t_0) and that $u(x_0, t_0) > 1$. Then $x_0 = 0$ or $x_0 = 1$. If $x_0 = 0$, then we have $u_x(x_0, t_0) = hu(x_0, t_0) > 0$. So near x_0, u is increasing, so there is a $\varepsilon > 0$ such that $u(\varepsilon, t_0) > u(x_0, t_0)$, a contradiction. If $x_0 = 1$, then we have $u_x(x_0, t_0) = -hu(x_0, t_0) < 0$. So near x_0, u is decreasing, so there is a $\varepsilon > 0$, such that $u(1 - \varepsilon, t_0) > u(1, t_0)$, a contradiction.

Therefore, the maximum of u on the boundary is 1, and by the maximum principle, we conclude that the minimum of u is non-negative and the maximum of u is 1. ∎

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(3.1)

Suppose that $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic (hence it is of class C^∞). Then for $1 \le i \le n$,

$$\Delta(u_{x_i}) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\left(\frac{\partial}{\partial x_i} u\right)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial^2}{\partial x_j^2} u\right). \quad \text{switching order of partials (ok since } u \in C^\infty(\Omega))$$

$$= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u\right)$$

$$= \frac{\partial}{\partial x_i}(\Delta u) = \frac{\partial}{\partial x_i}(0) = 0.$$

Hence u_{x_i} is harmonic.

By induction on k, if $u : \Omega \longrightarrow \mathbb{R}$ is harmonic, then its k^{th} order partial derivatives are all harmonic in Ω.

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(3.2) (a). Suppose u is subharmonic ($\Delta u \geq 0$). Letting $\sigma = x + r\sigma'$, we get $d\sigma = r^{n-1} d\sigma'$.

$$g(r) := \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(\sigma)\, d\sigma = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\sigma')\, d\sigma'$$

We calculate

$$g'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \nabla u(x + r\sigma') \cdot \sigma' \, d\sigma' \qquad \text{(Chain rule)}$$

$$= \frac{1}{\omega_n r} \int_{\partial B_1(0)} \nabla v(\sigma') \cdot \sigma' \, d\sigma' \quad \text{where } v(\sigma') = u(x + r\sigma')$$

$$= \frac{1}{\omega_n r} \int_{B_1(0)} \operatorname{div}(\nabla v)\, dy \qquad \text{(Divergence Theorem)}$$

$$= \frac{1}{\omega_n r} \int_{B_1(0)} \Delta v(y)\, dy$$

$$= \frac{r}{\omega_n} \int_{B_1(0)} \Delta u(y + r\sigma')\, dy \quad \text{since } \begin{matrix} \nabla v(\sigma') = r\nabla u(x + r\sigma') \\ \Delta v(\sigma') = r^2 \Delta u(x + r\sigma') \end{matrix}$$

$$\geq 0 \qquad (u \text{ subharmonic})$$

Thus $g(r)$ is increasing, and

$$g(r) \geq \lim_{r \to 0^+} g(r) = \lim_{r \to 0^+} \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(\sigma)\, d\sigma$$

$$= u(x) \lim_{r \to 0^+} \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} d\sigma = u(x).$$

giving

$$\boxed{u(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, dy.}$$

Integrate from 0 to R:

$$\omega_n r^{n-1} u(x) \leq \int_{\partial B_r(x)} u(y)\, dy$$

$$\Rightarrow \int_0^R \omega_n r^{n-1}\, dr \cdot u(x) \leq \int_0^R \int_{\partial B_r(x)} u(y)\, dy$$

$$\Rightarrow \frac{\omega_n R^n}{n} u(x) \leq \int_{B_R(x)} u(y)\, dy.$$

$$\Rightarrow \boxed{u(x) \leq \underbrace{\frac{n}{\omega_n R^n}}_{\text{vol}(B_R(x))} \int_{B_R(x)} u(y)\, dy}.$$

If u is superharmonic, all inequalities above are reversed.

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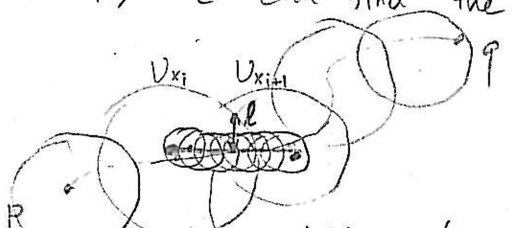
1(b) Suppose $u \in C(\overline{\Omega})$ is subharmonic and the maximum m is attained at $p \in \Omega$. We show that for any $q \in \Omega$, $u(q) = m$, i.e. u is constant on Ω.

Step 1. We find a sequence of circles $B(x_j) \subset \Omega$, $0 \le j \le N$, such that

- $x_j \in B(x_{j-1})$, $1 \le j \le N$,
- $x_0 = p$, $x_N = q$.

Since Ω is path-connected, there exists a continuous function $f: [0,1] \to \Omega$ such that $f(0) = p$ and $f(1) = q$. For each point $x \in \Omega$ there exists a ball U_x around x contained entirely in Ω, since Ω is open. Since f is continuous, each $f^{-1}(U_x)$ is open. These sets cover $[0,1]$ so by compactness of $[0,1]$ there is a finite subcover. Thus path f lies completely in the union of a finite number of U_x; so we can order them $U_{x_1}, U_{x_2}, \ldots$ such that $p \in U_{x_1}$, $q \in U_{x_m}$, and U_{x_i} intersects $U_{x_{i+1}}$. By adding balls centered on the segments joining x_i, x_{i+1}, we can find the desired balls.



→ Add circles of radius r spaced at a fixed distan less than r along this line, so that each circle contains the center of the next one. Choose $r < l$ so the circles fit in $U_{x_i} \cup U_{x_{i+1}} \subset \Omega$.

Step 2. We claim that if u attains the maximum m at $x \in \Omega$, and $B_R(x) \subset \Omega$, then $u(y) = m$ for every $y \in B(p)$.

Suppose by way of contradiction that $u(z) \neq m$ for some $z \in B_R(x)$. Then $u(z) < m$. Since u is continuous, there exists a circle $B_r(z)$ around z where $u < m$. Then

$$m = u(x) \overset{\text{(from (a))}}{\le} \frac{1}{\operatorname{vol} B_R(x)} \int_{B_R(x)} u(y)\,dy = \frac{1}{\operatorname{vol} B_R(x)} \left(\int_{B_R(x) - B_r(z)} u(y)\,dy + \int_{B_r(z) \cap B_R(x)} u(y)\,dy \right)$$

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(b) cont.

$$< \frac{1}{\mathrm{vol}\, B_R(x)} \left(m\left(\mathrm{vol}(B_R(x)) - \mathrm{vol}(B_r(z))\right) + m\, \mathrm{vol}(B_r(z) \cap B_R(x)) \right)$$ by maximality of $u(x) = m$; strict inequality since $u < m$ on $B_r(z)$

$$= m$$

contradiction.

<u>Step 3.</u> Taking the circles from Step 1, we show by induction that $u = m$ on $B(x_i)$. For $B(x_0)$ this is true by Step 2. If $u = m$ on $B(x_i)$, then since $x_{i+1} \in B(x_i)$, we get $u(x_{i+1}) = m$ and by Step 2, $u = m$ on $B(x_{i+1})$. Since $q = x_N \in B(x_N)$ we get $f(q) = m$.

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If $u \in C(\bar{\Omega})$ is superharmonic, then reverse all signs in the above argument and replace "maximum" by "minimum".

(c)

$$\begin{aligned} \Delta(u^2) &= \mathrm{div}(\nabla(u^2)) \\ &= \mathrm{div}(2u \nabla u) \quad \text{(Chain rule)} \\ &= 2\nabla u \cdot \nabla u + 2u \Delta u \quad \text{(Product rule)} \\ &= 2(|\nabla u|^2 + u \Delta u) \end{aligned}$$

If u is harmonic then $\Delta u = 0$ so

$$\Delta(u^2) = 2|\nabla u|^2 \geq 0$$

and u^2 is subharmonic.

(d).

$$\begin{aligned} \Delta(F(u)) &= \mathrm{div}(\nabla(F(u))) \\ &= \mathrm{div}[F'(u) \nabla u] \quad \text{(Chain rule)} \\ &= F''(u) \nabla u \cdot \nabla u + F'(u) \Delta u \quad \text{(Product rule)} \end{aligned}$$

$F \circ u$ is subharmonic if $F''(u(x))|\nabla u(x)|^2 + F'(u(x)) \Delta u(x) \geq 0$ for all $x \in \Omega$. In particular, since $\Delta u \geq 0$ and $|\nabla u|^2 \geq 0$, it is sufficient that $F'(x), F''(x) \geq 0$ for all $x \in u(\Omega)$.

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... are non-negative..

5. **(3.4)** Let B_R be the circle of radius R centered at $(0,0)$. Use the method of separation of variables to solve the problem

$$\begin{cases} \overset{\Delta U = f}{\cancel{\Delta u = 0}} & \text{in } B_R \\ u = 1 & \text{on } \partial B_R \end{cases}$$

Find an explicit formula when $f(x,y) = y$.

First consider the problem of finding a solution to

$$\Delta u = f,$$

in a circle with radius R, where f can be written as a Fourier sine series. We start by supposing that u can be written as $\sum v_k(r)w_k(\theta)$ where the w_k are eigenfunctions from the homogeneous problem. To find those, suppose that U is a solution to the homogenous problem, $\Delta U = 0$, and further suppose that U can be written as $U(r,\theta) = v(r)w(\theta)$. Then we know

$$v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0.$$

Simplifying, we see that

$$\frac{r^2v''(r) + rv'(r)}{v(r)} = \frac{-w''(\theta)}{w(\theta)}.$$

Since the left-hand side is a function of r alone and the right-hand side is a function of θ alone, we need both sides to equal some constant λ.

In particular, we have $w''(\theta)+\lambda w(\theta) = 0$. Since our domain is a circle, we require that $w(\theta) = w(\theta+2\pi)$. To satisfy the periodic conditions, we must have $\lambda = -\mu^2 < 0$, and $w(\theta) = A\sin(\mu\theta) + B\cos(\mu\theta)$. Since we have that f can be expressed as Fourier sine series, we need only consider $w(\theta) = A\sin(\mu\theta)$. Since $w(\theta) = w(\theta+2\pi)$, we need $\sin(\mu\theta) = \sin(\mu\theta + \mu 2\pi)$. So $\mu_k = \frac{k}{2}$, and we get $w_k(\theta) = A\sin\left(\frac{k}{2}\theta\right)$.

Now suppose that $f \doteq \sum b_k(r)\sin\left(\frac{k}{2}\theta\right)$. Then we want to find appropriate $v_k(r)$ such that $U = \sum v_k(r)\sin\left(\frac{k}{2}\theta\right)$ is a solution to $\Delta U = f$. Substituting this sum for U, we see that $v_k(r)$ must satisfy

$$v_k''(r) + \frac{1}{r}v_k'(r) \cancel{+} \frac{k^2}{4r^2}v_k(r) = b_k(r).$$

Now in the case that $f(x,y) = y$, we have $f(r,\theta) = r\sin(\theta)$. Notice that $f(r,\theta)$ is already in the form of a Fourier sine series (it consists of just one term, corresponding to $k = 2$.) So our solution will be of the form $v(r)\sin(\theta)$, where $v(r)$ satisfies

$$v''(r) + \frac{1}{r}v'(r) \cancel{+} \frac{1}{r^2}v(r) = r$$

If we let $s = \log r$, then our equation becomes

$$v''(s) - v(s) = e^{3s},$$

which has a solution $v(s) = \frac{1}{8}e^{3s} + C_1e^{s} + C_2e^{-s}$. In terms of the original variables, $v(r) = \frac{1}{8}r^3 + C_1r + C_2r^{-1}$. However, as argued in the book, we do not consider the r^{-1}.

So our solution is

$$u(r,\theta) = \frac{1}{8}r^3\sin(\theta) + \frac{1}{8}C_1r\sin(\theta)$$

We can find C_1 such that $u(R,\theta) = 0$ and then we would have $1+u$ as a solution to the original problem. Doing so, we get $u(r,\theta) = 1 + \frac{1}{8}\left(r^3 - R^2r\right)\sin(\theta)$. In rectangular coordinates, we get

$$u(x,y) = \frac{1}{8}\left(\left(x^2+y^2\right)y - R^2y\right) + 1$$

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