1.(a) By Problem 2.11, the probability distribution of the particle's position at time t is given by

$$p(x,t) = \Gamma_D(x,t) - \Gamma_D(x-2L,t)$$

(We assume that the particle stops when it first reaches L; this does not affect T_L.)

Now the probability that the particle hasn't reached L at time t is

$$\text{Prob}\{T_L > t\} \overset{\text{since particle stops when it reaches } L}{=} \text{Prob}\{-\infty < X(t) < L\}$$ ← probability particle is at position in $(-\infty, L)$ at time t.

$$= \int_{-\infty}^{L} p(x,t)\,dx$$

$$= \int_{-\infty}^{L} \Gamma_D(x,t) - \Gamma_D(x-2L,t)\,dx$$

$$= \int_{-\infty}^{L} \Gamma_D(x,t) - \int_{-\infty}^{-L} \Gamma_D(x,t)\,dx$$

$$= \int_{-L}^{L} \Gamma_D(x,t)\,dx. \checkmark \quad (1)$$

$$\leq 2L \max_{\substack{x\in\mathbb{R} \\ t \text{ fixed.}}} \Gamma_D(x,t).$$

$$= 2L\frac{1}{\sqrt{4\pi D t}} \checkmark$$

Since $\frac{2L}{\sqrt{4\pi D t}} \to 0$ as $t \to \infty$, we conclude

$$\lim_{t\to\infty} \text{Prob}\{T_L < t\} = 1 - \overbrace{\lim_{t\to\infty}\text{Prob}\{T_L > t\}}^{=0} = 1 \checkmark$$

so the particle reaches L in finite time with probability 1.

✓ 4/4

(b) From (1),

$$\text{Prob}\{T_L \le t\} = 1 - \text{Prob}\{T_L > t\}$$

$$= \boxed{1 - \int_{-L}^{L} \Gamma_D(x,t)\,dx} \checkmark$$

3/3

1.(c). The probability that the particle passes passes through $x=L$ for the first time in the time interval $(t, t+dt)$ is

$$P(\{T_L > t\}) - P(\{T_L \geq t+dt\})$$

$$= \int_{-L}^{L} \Gamma_D(x,t)\,dx - \int_{-L}^{L} \Gamma_D(x, t+dt)\,dx \quad \text{from (1).}$$

$$= \boxed{\left(\int_{-L}^{L} -\frac{d}{dt}(\Gamma_D(x,t))\,dx \right) \cdot dt}$$

$$= \left[\frac{d}{dt}\left(\int_{-L}^{L} \Gamma_D(x,t)\,dx\right)\right] dt.$$

$$= \left(\int_{-L}^{L} \frac{-1}{4\sqrt{\pi D t^3}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{8\sqrt{\pi D^3 t^5}} e^{-\frac{x^2}{4Dt}}\,dx \right) dt.$$

3/3

10/10

$$= dt \int_{-L} \frac{u}{dt}\Gamma_D(x,t)dx$$

$$= -dt \int_{-L}^{L} \left(\frac{-1}{4\sqrt{\pi D}} t^{\frac{-3}{2}} e^{\frac{-x^2}{4Dt}} + \frac{x^2}{8\sqrt{\pi D^3}} t^{\frac{-5}{2}} e^{\frac{-x^2}{4Dt}} \right) dx$$

2. Consider the problem

consulted · erman, "Applied Partial Differential Equations"

$$\begin{cases} u_t - u_{xx} = 0 & x > 0, t > 0 \\ u(x,0) = 0 & x \geq 0 \\ u_x(0,t) = g(t) & t > 0 \end{cases}$$

2/3

(a) Using Fourier (cosine) transform solve the problem above where g is continuous and bounded in $L^2(0,\infty)$. Prove that this is the only solution.

Let $\hat{u}$ denote the Fourier cosine transform of $u(x,t)$. That is,

$$\hat{u}(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\cos(\omega x)dx.$$

Taking the transform of the pde, we get

$$\frac{d}{dt}\hat{u}(\omega,t) = \frac{-2}{\pi}g(t) - \omega^2\widehat{u}(\hat{\omega},t)$$

Solving, we see that

$$\hat{u}(\omega,t) = \frac{-2}{\pi}e^{-\omega^2 t}\int_0^t g(y)e^{\omega^2 y}dy$$

So we have

$$u(x,t) = \frac{-2}{\pi}\int_0^\infty e^{-\omega^2 t}\cos(\omega x)\left(\int_0^t g(y)e^{\omega^2 y}dy\right)d\omega.$$

To prove that this is unique, we first will extend the domain from $x > 0$ to $x \in \mathbb{R}$ through even reflection. So if there is another v that satisfies

$$\begin{cases} v_t - v_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_x(0,t) = g(t) & t > 0 \end{cases}$$

(note, the even reflection guarantees that $v_x(0,t) = 0$ because v would be symmetric about $x = 0$.) then $w = u - v$ satisfies

$$\begin{cases} w_t - w_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ w(x,0) = 0 & x \in \mathbb{R} \\ w_x(0,t) = 0 & t > 0 \end{cases}$$

which has unique solution $w = 0$. So $u = v$ is the unique solution.

(b) Prove that, without the condition that g is in $L^2(0,\infty)$, the problem above does not have a unique solution by using the two functions $w_1(x,t) = e^x \sin(2t+x)$ and $w_2(x,t) = -e^{-x}\sin(2t-x)$. Notice that

$$\begin{aligned} w_{1t} &= 2e^x \cos(2t+x) \\ w_{1x} &= e^x \sin(2t+x) + e^x \cos(2t+x) \\ w_{1xx} &= 2e^x \cos(2t+x) \end{aligned}$$

and

$$\begin{aligned} w_{2t} &= -2e^{-x} \cos(2t-x) \\ w_{2x} &= e^{-x} \sin(2t-x) + e^{-x} \cos(2t-x) \\ w_{2xx} &= -2e^{-x} \cos(2t-x) \end{aligned}$$

Notice that both w_1 and w_2 satisfy $u_t - u_{xx} = 0$ and $u_x(0,t) = 0$. However, $w_1(x,0) = e^x \sin(x)$ and $w_2(x,0) = e^{-x}\sin(x)$.

We know a solution to

$$\begin{cases} u_x - u_{tt} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = e^{|x|}\sin(|x|) & x \in \mathbb{R} \\ u_x(0,t) = 0 & t > 0 \end{cases}$$

is given by

$$\begin{aligned} u_1(x,t) &= \int_{\mathbb{R}} e^{|y|} \sin(|y|)\Gamma(x-y,t)dy \\ &= \int_0^\infty e^y \sin(y)\Gamma(x-y,t)dy + \int_{-\infty}^0 e^{-y}\sin(-y)\Gamma(x-y,t)dy \\ &= \int_0^\infty e^y \sin(y)\Gamma(x-y,t)dy + \int_\infty^0 e^w \sin(w)\Gamma(x+w,t)dw \end{aligned}$$

where $w = -y$. We can rewrite this as

$$u_1(x,t) = \int_0^\infty e^y \sin(y)\left(\Gamma(x-y,t) - \Gamma(x+y,t)\right) dy$$

(Note, we extended the domain from $x > 0$ to $x \in \mathbb{R}$, and adjusted the initial condition through even reflection. This makes u_1 symmetric around 0, so indeed $u_{1x}(0,t) = 0$.)

Similarly, a solution to

$$\begin{cases} u_x - u_{tt} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = -e^{-|x|}\sin(|x|) & x \in \mathbb{R} \\ u_x(0,t) = 0 & t > 0 \end{cases}$$

is given by

$$u_2(x,t) = \int_0^\infty e^{-y} \sin(y)\left(\Gamma(x-y,t) - \Gamma(x+y,t)\right) dy$$

(Note, we have again extended the domain from $x > 0$ to $x \in \mathbb{R}$ via even reflection).

So the function $z_1 = w_1 - u_1$ satisfies

$$\begin{cases} z_t - z_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ z(x,0) = 0 & x \in \mathbb{R} \\ z_x(0,t) = \sin(2t) + \cos(2t) & t > 0 \end{cases}$$

Similarly, the function $z_2 = w_2 - u_2$ satisfies

$$\begin{cases} z_t - z_{xx} = 0 & x \in \mathbb{R} t > 0 \\ z(x,0) = 0 & x \in \mathbb{R} \\ z_x(0,t) = \sin(2t) + \cos(2t) & t > 0 \end{cases}$$

Notice that

$$\int_0^L (\sin(2t) + \cos(2t))^2 dt = L + \int_0^L 2\sin(2t)\cos(2t) dt,$$

which goes to ∞ as $L \to \infty$. So the function $\sin(2t) + \cos(2t)$ is not L^2.
We claim that $z_1 \neq z_2$. Consider $z_1(0,t)$ and $z_2(0,t)$.

$$z_1(0,t) = \sin(2t) - \int_0^\infty e^y \sin(y)\left(\Gamma(-y) - \Gamma(y)\right) dy = \sin(2t)$$

since $\Gamma(-y) = \Gamma(y)$, Similarly,

$$z_2(0,t) = -\sin(2t).$$

Since $z_1(0,t) \neq z_2(0,t)$, we see that $z_1 \neq z_2$. Great Job!

3. Let $v(x,t) = u(x,t)e^{\cdots}$ 10/10

Then $v_t = u_t e^{hx+kt} + ku e^{hx+kt}$

$v_x = u_x e^{hx+kt} + hu e^{hx+kt}$

$v_{xx} = u_{xx} e^{hx+kt} + h u_x e^{hx+kt} + h u_x e^{hx+kt} + h^2 u e^{hx+kt}$

$v_t - Dv_{xx} = e^{hx+kt}(u_t + ku - Du_{xx} - 2hDu_x - h^2 Du)$ ✓

To make $v_t - Dv_{xx} = 0$, we need $2hD = b$ and $h^2 D - k = c$. ✓

Thus $h = \frac{b}{2D}$ ✓ and $k = \frac{b^2}{4D} - c$ ✓

If h and k are set as above, then

$$\begin{cases} v_t - Dv_{xx} = 0 \\ v(x,0) = u(x,0)e^{hx} = g(x)e^{bx/2D} \triangleq G(x) \end{cases}$$ ✓

$$\Rightarrow v(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} g(y) e^{\frac{by}{2D}} e^{-\frac{(x-y)^2}{4Dt}} dy.$$

$$\Rightarrow u(x,t) = v(x,t)e^{-hx-kt} = v(x,t)e^{-\frac{b}{2D}x - (\frac{b^2}{4D}-c)t}$$

$$= \frac{e^{-\frac{b}{2D}x - (\frac{b^2}{4D}-c)t}}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} g(y) e^{\frac{by}{2D} - \frac{(x-y)^2}{4Dt}} dy$$ ✓

If $c<0$, g is bounded, then assume that $|g(y)| \le M$, $\forall y \in \mathbb{R}$

Thus $$u(x,t) \le e^{-\frac{b}{2D}x - (\frac{b^2}{4D}-c)t} \frac{M}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{\frac{by}{2D} - \frac{(x-y)^2}{4Dt}} dy \quad (**)$$

Note that when t, x are fixed, $$\frac{1}{\sqrt{4Dt}} \int_{\mathbb{R}} e^{\frac{by}{2D} - \frac{(x-y)^2}{4Dt}} dy \quad (*)$$

$= \int_{\mathbb{R}} e^{-z^2+d} dz$, where $z = \frac{y}{\sqrt{4Dt}} + w$, w and d are constants determined by b, D and t. Thus $(*)$ is bounded. i.e. $(*) < K$, $\forall t$ for some constant K. ✓ ✓ Great!

Back to $(**)$. Since $\frac{b^2}{4D} - c > 0$, $u(x,t) \to 0$ as $t \to +\infty$. ✓ 4

4 Problem 2.15

Find an explicit formula for the solution of the Cauchy problem

$$\begin{cases} u_t = u_{xx} & x > 0, t > 0 \\ u(x,0) = g(x) & x \geq 0 \\ u(0,t) = 0 & t > 0. \end{cases}$$

with g continuous and $g(0) = 0$.
Hint: Extend g to $x < 0$ by odd reflection: $g(-x) = -g(x)$. Solve the corresponding global Cauchy problem and write the result as an integral on $(0, +\infty)$.

Solution: Following the hint, consider the global Cauchy problem $v_t = v_{xx}$ on $\mathbb{R} \times (0, \infty)$ with initial condition $v(x,0) = h(x)$ where $h(x) = g(x)$ for $x > 0$ and $h(x) = -g(|x|)$ for $x < 0$, and $h(0) = g(0) = 0$. The text gives us the solution

$$v(x,t) = \int_{\mathbb{R}} h(y)\Gamma_D(x-y,t)dy.$$

Since on the interval $[0,\infty) \times [0,\infty)$ the problems in u and v are the same, this function v solves the first two parts of this Cauchy problem. We just need to check that $v(0,t) = 0$. Observe that

$$\begin{aligned} v(0,t) &= \int_{\mathbb{R}} h(y)\Gamma_D(0-y,t)dy \\ &= \int_{-\infty}^{0} h(y)\Gamma_D(-y,t)dy + \int_{0}^{\infty} h(y)\Gamma_D(-y,t)dy \\ &= -\int_{\infty}^{0} h(-z)\Gamma_D(-(-z),t)dz + \int_{0}^{\infty} h(y)\Gamma_D(-y,t)dy && \text{by the change of vars } y \to -z \\ &= \int_{0}^{\infty} -h(z)\Gamma_D(z,t)dz + \int_{0}^{\infty} h(y)\Gamma_D(y,t)dy && \text{since } \Gamma_D(-y,t) = \Gamma_D(y,t) \\ &= \int_{0}^{\infty} h(y)\Gamma_D(y,t) - h(y)\Gamma_D(y,t)dy = 0 && \text{by the change of vars } z \to y. \end{aligned}$$

Thus we have $v(0,t) = 0$. Finally, we need to write $u = v$ as an integral over $(0, \infty)$. Following the same steps as above, we can turn $u = v$ into the integral

$$u(x,t) = \int_{0}^{\infty} g(y)\left(\Gamma_D(x-y,t) - \Gamma_D(x+y,t)\right)dy$$

and we are done.

10/10

5 Problem 2.16

Let $Q_T = \Omega \times (0,T)$, with Ω bounded domain in $\mathbb{R}^n$. Let $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfy the equation

$$u_t = D\Delta u + \mathbf{b}(\mathbf{x},t) \cdot \nabla u + c(\mathbf{x},t)u \tag{2}$$

5. (2.16) Let $Q_T = \Omega \times (0,T)$ with Ω bounded domain in $\mathbb{R}^n$. Let $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfy the equation

$$u_t = D\Delta u + \mathbf{b}(\mathbf{x},t) \cdot \nabla u + c(\mathbf{x},t)u$$

in Q_T, where $\mathbf{b}$ and c are continuous in $\overline{Q_T}$. Show that if $u \geq 0$ (resp. $u \leq 0$) on $\partial_p Q_T$ then $u \geq 0$ (resp. $u \leq 0$) in Q_T.

Proof. We will first prove that if $u \geq 0$ on $\partial_p Q_T$ then $u \geq 0$ in Q_T. To this end, we first simplify our consideration to the case that $c(\mathbf{x},t) \leq a < 0$. So suppose that $u(\mathbf{x},t) = v(\mathbf{x},t)e^{kt}$. Then $u_t = D\Delta u + \mathbf{b}(\mathbf{x},t) \cdot \nabla u + c(\mathbf{x},t)u$ if and only if

$$v_t(\mathbf{x},t)e^{kt} = e^{kt}\left(D\Delta u + \mathbf{b}(\mathbf{x},t) \cdot \nabla v + \left(c(\mathbf{x},t) - k\right)v\right)$$

Since $\overline{Q_T}$ is compact, we know that c is bounded above by some M. So if we choose $k > M$, then v satisfies

$$v_t(\mathbf{x},t) = D\Delta u + \mathbf{b}(\mathbf{x},t) \cdot \nabla v + c^*(\mathbf{x},t)v$$

where $c^*(\mathbf{x},t) = c(\mathbf{x},t) - k \leq M - k < 0$. Since $u \geq 0$ iff $v \geq 0$, we need only consider the case where $c(\mathbf{x},t) \leq a < 0$, (as we can simplify the problem to this case through the above procedure, choosing an appropriate k).

Now, we know that $u \in C^{2,1}(\overline{Q_T})$. Since $\overline{Q_T}$ is compact, u achieves a minimum at some point $(\mathbf{x_0},t_0)$. We will show that $u(\mathbf{x_0},t_0) \geq 0$. There are three possibilites: $(\mathbf{x_0},t_0) \in \partial_p Q_T$, $(\mathbf{x_0},t_0) \in Q \times (0,T)$, and $(\mathbf{x_0},t_0) \in Q \times \{T\}$.

Case 1: $(\mathbf{x_0},t_0) \in \partial_p Q_T$

In this case, we know that $u(\mathbf{x_0},t_0) \geq 0$, by hypothesis.

Case 2: $(\mathbf{x_0},t_0) \in Q \times (0,T)$

In this case, we know that $\nabla u(\mathbf{x_0},t_0) = 0$, by the first partial derivative test. Furthermore, since $u(\mathbf{x_0},t_0)$ is a minimum, we know that $\Delta u(\mathbf{x_0},t_0) \geq 0$, by the second partial derivative test. We also know that $u(x_0,t_0)$ is a minimum with respect to time, so we have $u_t(x_0,t_0) = 0$. So we have

$$c(x_0,t_0)u(x_0,t_0) = u_t(x_0,t_0) - D\Delta u(x_0,t_0) \leq 0$$

Since $c(\mathbf{x_0},t_0) < 0$, we must have $u(x_0,t_0) \geq 0$.

Case 3: $(x_0,t_0) \in Q \times \{T\}$.

In this case, we have $t_0 = T$ and we must have $u_t(x_0, t_0) \leq 0$, since the function must decrease to $u(x_0, T)$. Like above, we have $\Delta u(x_0, t_0) \geq 0$ and $\nabla u(x_0, t_0) = 0$. So

$$c(\mathbf{x}_0, t_0)u(\mathbf{x}_0, t_0) = u_t(\mathbf{x}_0, t_0) - D\Delta u(\mathbf{x}_0, t_0) \leq 0.$$

Since $c(\mathbf{x}, t) < 0$, we conclude that $u(\mathbf{x}_0, t_0) \geq 0$.

So in all three cases, we see that $u(\mathbf{x}_0, t_0)$ is non-negative. That is,

$$min_{\overline{Q_T}} u(\mathbf{x}, t) \geq 0,$$

and we conclude that $u \geq 0$ in Q_T.

$\square$

If $u \leq 0$ on $\partial_p Q_T$, we could apply the same ideas above but with the function $w = -u$ and would conclude that on Q_T, $w \geq 0$ so $u \leq 0$.

10/10 Very Nice!