18.152 Fall 2010 – Practice problems for the midterm

The exam will cover material from the following pages of the book: 1–58, 68–76, 90–96, 102–117.

1. (a) Let u be the solution to

$$u_t = Du_{xx} + bu_x + cu.$$

Find h and k such that $v(x,t) = u(x,t)e^{hx+kt}$ solves

$$v_t = Dv_{xx}.$$

(b) Solve the following Cauchy-Dirichlet problem using part (a) and separation of variables:

$$\begin{cases} u_t = u_{xx} + mu + \sin 2\pi x + 2\sin 3\pi x & 0 < x < 1,\ 0 < t \\ u(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & 0 < t. \end{cases}$$

2. Consider the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 1 & 0 < x < 1,\ 0 < t \\ u(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & 0 < t. \end{cases}$$

(a) Find the stationary ($u_t = 0$) solution $u^s(x) = u(x,t)$ for this problem.
(b) Prove that $u(x,t) \le u^s(x)$ for $t > 0$.
(c) Find $\beta > 0$ such that

$$u(x,t) \ge (1 - e^{-\beta t})u^s(x).$$

(d) Deduce that $u(x,t) \to u^s(x)$ as $t \to \infty$ uniformly in $[0,1]$.

3. Let u be solution to

$$\begin{cases} u_t(x,t) - Du_{xx}(x,t) = 0 & 0 < x < L,\ 0 < t \\ u(x,0) = g(x) & 0 \le x \le L \\ -u_x(0,t) = u_x(L,t) = 0 & 0 < t. \end{cases}$$

(a) Interpret the problem by assuming that $u(x,t)$ is the concentration of a gas at position x at time t. Justify intuitively the fact that

$$u(x,t) \to U, \qquad \text{as } t \to \infty,$$

where U is a constant. By integrating in an appropriate way the equation in the problem above find the value of U in terms of an integral involving g.

(b) Assume now that g belongs to $C([0,L])$ and u to $C([0,L]\times[0,\infty)) \cap C^1([0,L]\times[t_0,\infty))$ for all $t_0 > 0$. Prove that

$$\lim_{t\to\infty} \int_0^L (u(x,t) - U)^2 dx = 0.$$

Hint: Set $w(x,t) = u(x,t) - U$, define $E(t) = \int_0^L w^2(x,t)dx$, prove that

$$E'(t) \le -\frac{2D}{L}E(t)$$

and deduce from here an upper bound for $E(t)$ that converges to zero.

4. Assume that u is harmonic in $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Prove that

(a) for any v in $C^1(\overline{\Omega})$

$$\int_{\partial\Omega} v \partial_\nu u(\sigma)\, d\sigma = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

(b) and

$$\int_{\partial\Omega} \partial_\nu u(\sigma)\, d\sigma = 0.$$

5. Consider the ring

$$C_{1,R} = \{(r,\theta) \quad / \quad 1 < r < R, \;\; 0 \le \theta \le 2\pi\}.$$

(a) Given g and h in $C^1(\mathbb{R})$, 2π-periodic functions, find the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \; C_{1,R} \\ u(1,\theta) = g(\theta) & 0 \le \theta \le 2\pi \\ u(R,\theta) = h(\theta) & 0 \le \theta \le 2\pi. \end{cases}$$

(b) Write the formula when $h(\theta) = 1$ and $g(\theta) = \sin\theta$.