18.152 Practice Final Exam Solutions

(Corrected version)

① a) Let us take $v(x,t) := u(x,t) - \frac{x}{\pi}$, and $h(x) := g(x) - \frac{x}{\pi}$

Then, v solves:

$$(*) \begin{cases} v_t - Dv_{xx} = 0 \\ v(x,0) = h(x) \\ v_x(0,t) = 0, \ v(\pi,t) = 0 \end{cases}$$

$$v_k(x,t) = e^{-D(k+\frac{1}{2})^2 t} \cos\left((k+\tfrac{1}{2})x\right)$$

We want:

$$v = \sum_{k=0}^{+\infty} a_k e^{-D(k+\frac{1}{2})^2 t} \cos\left((k+\tfrac{1}{2})x\right), \text{ so } \boxed{u(x,t) = \frac{x}{\pi} + \sum_{k=0}^{+\infty} a_k e^{-D(k+\frac{1}{2})^2 t} \cos\left((k+\tfrac{1}{2})x\right)}$$

• So, we need $\sum_{k=0}^{+\infty} a_k \cos\left((k+\frac{1}{2})x\right) = h(x)$

The condition we need to find such a_k is $\underline{h'(0) = 0, \ h(\pi) = 0}$ (so we should modify the assumptions a little bit).

The a_k are then found by extending h to an even 4π-periodic function, and expanding the obtained function into a cosine series.

b) As for uniqueness, it suffices to show that in $(*)$, $h \equiv 0 \Rightarrow v \equiv 0$.

Namely, if u_1 and u_2 solve the original equation, then $\tilde{v} := u_1 - u_2$ solves $(*)$, $h \equiv 0$

Then $\frac{d}{dt} \int_0^\pi (\tilde{v}(x,t))^2 dx = 2 \int_0^\pi \tilde{v}_t(x,t) \tilde{v}(x,t) dx =$

$$= 2D \int_0^\pi \tilde{v}_{xx}(x,t) v(x,t) dx = -2D \int_0^\pi (\tilde{v}_x(x,t))^2 dx + \left(\tilde{v}_x(x,t) v(x,t)\right)\Big|_{x=0}^{x=\pi}$$

$$= -2D \int_0^\pi (\tilde{v}_x(x,t))^2 dx \leq 0$$

Since: $\int_0^\pi (\tilde{v}(x,0))^2 dx = \int_0^\pi (h(x))^2 dx = 0$, it follows that $v \equiv 0$

Hence $\tilde{v} \equiv 0$ so $u_1 \equiv u_2$. Uniqueness follows.

(2) a) The stationary u^s satisfies $-u^s_{xx} = x$

$$\Rightarrow u^s(x) = -\frac{1}{6}x^3 + Ax + B$$

$$u^s(0) = 0 \Rightarrow B = 0$$

$$u^s(1) = 0 \Rightarrow B = \frac{1}{6}$$

So:

$$\boxed{u^s(x) = -\frac{1}{6}x^3 + \frac{1}{6}x}$$

b) We note that $u^s(x) = \frac{1}{6}x(1-x^2) \geq 0$ for $0 \leq x \leq 1$

now, if $w(x,t) := u^s(x) - u(x,t)$, then w satisfies:

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) = 0 \\ \quad w(x,0) \geq 0 \end{cases}$$

Hence, by the Minimum Principle, it follows that $w \geq 0$, and so: $u(x,t) \leq u^s(x)$, for $t > 0$.

c) Let us take $f(t) := 1 - e^{-Bt}$ for some $B > 0$. Observe: $\lim_{t \to \infty} f(t) = 1$.

Let $w(x,t) := u(x,t) - f(t)u^s(x) = u(x,t) - u^s(x) + e^{-Bt}u^s(x)$

Then: $w_t - Dw_{xx} = \underbrace{(u_t - Du_{xx})}_{=x} + \underbrace{Du^s_{xx}}_{=-x} - Be^{-Bt}u^s(x) - e^{-Bt}Du^s_{xx}$

$$= e^{-Bt}\left(B \cdot \left(\frac{1}{6}x^3 - \frac{1}{6}x\right) + x\right) = e^{-Bt} \cdot \left(\frac{B}{6}x^3 + \left(1 - \frac{B}{6}\right)x\right)$$

Let us take $B = 6$. Then:

$$w_t - Dw_{xx} = e^{-Bt} \cdot \frac{B}{6}x^3 \geq 0$$

On the other hand, $w(x,0) = u(x,0) - f(0) \cdot u^s(x) = 0 - 0 = 0$

Hence, again by the Minimum Principle:

$$w \geq 0 \; ; \text{ so } u(x,t) \geq f(t)u^s(x)$$

d) From b) and c), and the fact that $\lim_{t \to \infty} f(t) = 1$, it follows that $u(x,t) \to u^s(x)$, as $t \to \infty$ uniformly in $[0,1]$.

(3) We can assume, without loss of generality, that the ball is centered at the origin.
The temperature depends only on the distance from the point to the origin. In other words,
$u(x,t) = u(r,t)$, where $r := |x|$.
We then use the formula for the Laplace operator in polar coordinates to deduce:
$$u_t - \Delta u = u_t - \left(u_{rr} + \frac{2}{r} u_r\right) = 0 ;\quad 0 < r < R,\ t > 0.$$
The additional conditions are:
$$\begin{cases} u(r,0) = \mathcal{U} ; & 0 \le r < R \\ u(R,t) = 0, & t > 0 \\ u \text{ is finite at } r = 0 ; & t > 0 \end{cases}$$
Let us note that: $u_{rr} + \frac{2}{r} u_r = \frac{1}{r}(ru)_{rr}$

Let $v := ru$
Then, the PDE is reduced to:
$$\begin{cases} v_t - v_{rr} = 0 ;\ 0 < r < R,\ t > 0 \\ v(r,0) = r\mathcal{U} ;\ 0 \le r < R \\ v(R,t) = v(0,t) = 0 ;\ t > 0 \end{cases}$$
We solve this problem by Separation of Variables:
$$v = \sum_{k=1}^{+\infty} a_k \cdot e^{-\frac{k^2\pi^2}{R^2} \cdot t} \cdot \sin\left(\frac{k\pi}{R} \cdot r\right)$$

$$a_k = \frac{2}{R} \int_0^R r\,\mathcal{U} \sin\left(\frac{k\pi}{R} \cdot r\right) dr = \frac{2R\mathcal{U}}{\pi k} \cdot (-1)^{k+1}$$
So: $$v = \sum_{k=1}^{+\infty} \frac{2R\mathcal{U}}{\pi k} \cdot (-1)^{k+1} e^{-\frac{k^2\pi^2}{R^2} \cdot t} \cdot \sin\left(\frac{k\pi}{R} \cdot r\right)$$
$$\Rightarrow u(r,t) = \frac{2R\mathcal{U}}{\pi r} \cdot \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} \cdot e^{\frac{-k^2\pi^2}{R^2} \cdot t} \cdot \sin\left(\frac{k\pi}{R} \cdot r\right) \quad (*)$$
We know that $u(0,t) \ge 0$ by the <u>Minimum Principle</u>.
On the other hand, we can let $r \to 0$ in $(*)$ to deduce:
$$\left\{ \frac{\sin x}{x} \to 1 \text{ as } x \to 0 \right\}$$

$$0 \le u(0,t) = 2\mathcal{U} \sum_{k=1}^{+\infty} (-1)^{k+1} e^{-\frac{k^2\pi^2}{R^2} t} \le 2\mathcal{U} e^{-\frac{\pi^2}{R^2} t}$$

by the alternating series test. So $u(0,t) \to 0$ exponentially fast.

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4 We use Green's Identity: For fixed $x \in \Omega$

$$\int_\Omega (G(x,y)\Delta_y u(y) - u(y)\Delta_y G(x,y))\,dy =$$

$$= \int_{\partial\Omega} (G(x,\sigma)\partial_\nu u(\sigma) - u(\sigma)\partial_\nu G(x,\sigma))\,d\sigma$$

Now: $G(x,\sigma) = 0$, $\Delta_y G(x,y) = -\delta_x(y)$, $\Delta_y u(y) = f(y)$

$$\Rightarrow \int_\Omega u(y)\,\delta_x(y)\,dy = -\int_\Omega G(x,y)f(y)\,dy - \int_{\partial\Omega} h(\sigma)\partial_\nu G(x,\sigma)\,d\sigma$$

$$\Rightarrow u(x) = -\int_\Omega G(x,y)f(y)\,dy - \int_{\partial\Omega} h(\sigma)\partial_\nu G(x,\sigma)\,d\sigma$$

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5 We use the <u>Maximum Principle</u> for the Laplace Equation to deduce that the maximum and minimum of u on the closure of B_1 are attained on ∂B_1.

We note that on ∂B_1: $x^4 + y^4 + z^4 \le x^2 + y^2 + z^2 = 1$

Equality is achieved for $(x,y,z) = (1,0,0)$, for example.

Consequently, $\boxed{\max_{\overline{B}_1} u = 1}$.

On the other hand,

Arithmetic Mean - Quadratic Mean Inequality

On ∂B_1: $x^4 + y^4 + z^4 = (x^2)^2 + (y^2)^2 + (z^2)^2 \ge 3 \cdot \left(\frac{x^2+y^2+z^2}{3}\right)^2 =$

$$= 3 \cdot \frac{1}{9} = \frac{1}{3}$$

Equality is achieved for $(x,y,z) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, for example.

Consequently, $\boxed{\min_{\overline{B}_1} u = \frac{1}{3}}$.

$$C_1(t) = -\frac{1}{k\ell} \int_0^t \sin(k\ell\tau) f_k(\tau)\,d\tau$$

$$C_2(t) = \frac{1}{k\ell} \int_0^t \cos(k\ell\tau) f_k(\tau)\,d\tau$$

So: $$w_k(t) = -\frac{1}{k\ell}\cos(k\ell t) \int_0^t \sin(k\ell\tau) f_k(\tau)\,d\tau + \frac{1}{k\ell}\sin(k\ell t)\int_0^t \cos(k\ell\tau) f_k(\tau)\,d\tau =$$

$$= \frac{1}{k\ell}\int_0^t \sin(k\ell(t-\tau)) f_k(\tau)\,d\tau =$$

$$= \frac{1}{k\ell}\int_0^t \sin(k\ell\tau) f_k(t-\tau)\,d\tau$$

Now, $f_k(t-\tau) = \frac{2}{L} \cdot \int_0^L f(\xi, t-\tau) \sin\left(\frac{k\xi\pi}{L}\right) d\xi$

So: $$u(x,t) = \frac{2}{c\pi} \sum_{k=1}^{+\infty} \frac{1}{k} \sin(k\ell t) \int_0^t \int_0^L f(\xi, t-\tau) \sin\left(\frac{k\xi\pi}{L}\right) \sin(k\ell\tau)\, d\xi\, d\tau$$

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(1)

- $f(x,t) = e^{-t}\sin\left(\frac{\pi x}{L}\right)$

$f_1(t) = e^{-t}$ here; so: $u(x,t) = w_1(t)\sin\left(\frac{\pi x}{L}\right)$, for

$$w_1(t) = \frac{L}{c\pi} \cdot \int_0^t \sin\left(\tau \cdot \frac{c\pi}{L}\right) e^{-(t-\tau)}\, d\tau$$

We calculate $\int_0^t \sin\left(\tau \cdot \frac{c\pi}{L}\right) e^{-(t-\tau)}\, d\tau$ by integrating by parts twice. A calculation then shows that:

$$u(x,t) = \frac{L^2}{L^2+c^2\pi^2} \cdot \left(e^{-t} - \cos\left(\frac{c\pi}{L}t\right) + \frac{L}{c\pi}\sin\left(\frac{c\pi}{L}t\right)\right)\sin\left(\frac{\pi x}{L}\right)$$

The calculation for $f(x) = xe^{-t}$ is similar, but more involved.

Computing f_k, one can show:

$$u(x,t) = \frac{2L}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1} L^2}{k(L^2+c^2\pi^2k^2)} \left(e^{-t} - \cos\left(\frac{c\pi}{L}t\right) + \frac{L}{\pi}\sin\left(\frac{c\pi}{t}L\right)\right)\sin\left(\frac{k\pi}{L}x\right)$$

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⑥ Suppose that $\Delta u = 0$

Then, if we take $\frac{\partial}{\partial x_j}$; $j = 1,2,3$, it follows that:

$$\Delta u_{x_j} = 0.$$

We write this as: $\Delta(\nabla u) = 0$ (where Δ acts componentwise)

Let us fix $x_0 \in \mathbb{R}^3$. We can use the Mean-Value Property (componentwise) to deduce that for all $R>0$:

$$\nabla u(x_0) = \frac{1}{\frac{4\pi}{3}R^3} \int_{B(x_0,R)} \nabla u(x)\, dx$$

Hence, by the Triangle Inequality and the Cauchy-Schwarz Inequality:

$$|\nabla u(x_0)| \leq \frac{1}{\frac{4\pi}{3}R^3} \cdot \int_{B(x_0,R)} |\nabla u(x)|\, dx$$

$$\leq \frac{1}{\frac{4\pi}{3}R^3} \cdot \left(\int_{B(x_0,R)} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_{B(x_0,R)} 1^2 dx\right)^{\frac{1}{2}} =$$

$$= \frac{1}{\left(\frac{4\pi}{3}R^3\right)^{\frac{1}{2}}} \cdot \left(\int_{B(x_0,R)} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\left(\frac{4\pi}{3}R^3\right)^{\frac{1}{2}}} \cdot \left(\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \to 0 \quad \text{as } R \to +\infty.$$

Consequently, $\nabla u \equiv 0$, so u is constant.

(7) We separate variables:

Let us write: $u(x,t) = \sum_{k=1}^{+\infty} w_k(t) \sin\left(\frac{k\pi}{L} \cdot x\right)$

$$f(x,t) = \sum_{k=1}^{+\infty} f_k(t) \sin\left(\frac{k\pi}{L} \cdot x\right)$$

We obtain: $\sum_{k=1}^{+\infty} \left(w_k''(t) + c^2 \cdot \frac{k^2\pi^2}{L^2} w_k(t) \right) \sin\left(\frac{k\pi}{L} \cdot x\right) = \sum_{k=1}^{+\infty} f_k(t) \sin\left(\frac{k\pi}{L} \cdot x\right)$

So:
$$\begin{cases} w_k''(t) + c^2 \frac{k^2\pi^2}{L^2} w_k(t) = f_k(t) \\ w_k(0) = 0, \ w_k'(0) = 0 \end{cases} ; k \in \mathbb{N}$$

Let $l := \frac{c\pi}{L}$

$$(*) \begin{cases} w_k''(t) + k^2 l^2 w_k(t) = f_k(t) \\ w_k(0) = 0, \ w_k'(0) = 0 \end{cases}$$

We solve $(*)$ by Variation of Parameters:

The solution to the homogeneous equation is
$\tilde{w}_k(t) = c_1 \cos(klt) + c_2 \sin(klt)$.

Hence, we look for
$w_k(t) = c_1(t) \cos(klt) + c_2(t) \sin(klt)$,

With the additional assumption:
$c_1'(t) \cos(klt) + c_2'(t) \sin(klt) = 0$

We have to solve the system:

$$\begin{cases} c_1'(t) \cos(klt) + c_2'(t) \sin(klt) = 0 \\ -kl\, c_1'(t) \sin(klt) + kl\, c_2'(t) \cos(klt) = f_k(t) \end{cases}$$

We can check that the solution to this system is given by:

$$c_1'(t) = -\frac{1}{kl} \sin(klt) \cdot f_k(t), \quad c_2'(t) = \frac{1}{kl} \cos(klt) \cdot f_k(t)$$

Since $c_1(0) = c_2(0) = 0$, it follows that:

(8) We recall (see Page 280 in the textbook) that the unique solution $w \in C^2(\mathbb{R}^2 \times [0,+\infty))$ of the problem:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \ ; \ x \in \mathbb{R}^2, t>0 \\ w(x,0) = 0, w_t(x,0) = h(x) \ ; \ x \in \mathbb{R}^2 \end{cases}$$

is given by:

$$w(x,t) = \frac{1}{2\pi c} \cdot \int_{B_{ct}(x)} \frac{h(y)\,dy}{\sqrt{c^2t^2 - |x-y|^2}}.$$

The claim now follows from Duhamel's Principle.

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(9) a) We are looking at $u_{tt} - t u_{xx} = 0$.

The associated discriminant is $4t$ $\to$ <u>HYPERBOLIC</u> for $t>0$; <u>PARABOLIC</u> for $t=0$; <u>ELLIPTIC</u> for $t<0$

b) $\partial_t^2 - t\,\partial_x^2$ factorizes as:

$$(\partial_t - \sqrt{t}\,\partial_x)(\partial_t + \sqrt{t}\,\partial_x).$$

We then look at:

$$\varphi_t - \sqrt{t}\,\varphi_x = 0, \quad \psi_t + \sqrt{t}\,\psi_x = 0$$

$$\Rightarrow \varphi = F(3x + 2t^{\frac{3}{2}}), \quad \psi = G(3x - 2t^{\frac{3}{2}})$$

So, the characteristics are given by $3x \pm 2t^{\frac{3}{2}} = \text{const}$.

c) We set $\xi := 3x + 2t^{\frac{3}{2}}$, $\eta := 3x - 2t^{\frac{3}{2}}$

So, the canonical form is:

$$u_{\xi\eta} = 0$$

Thus, the general solution to the Tricomi Equation is:

$$u(x,t) = F(3x + 2t^{\frac{3}{2}}) + G(3x - 2t^{\frac{3}{2}}) \ ; \ F, G \text{ - arbitrary } C^2 \text{ functions}$$

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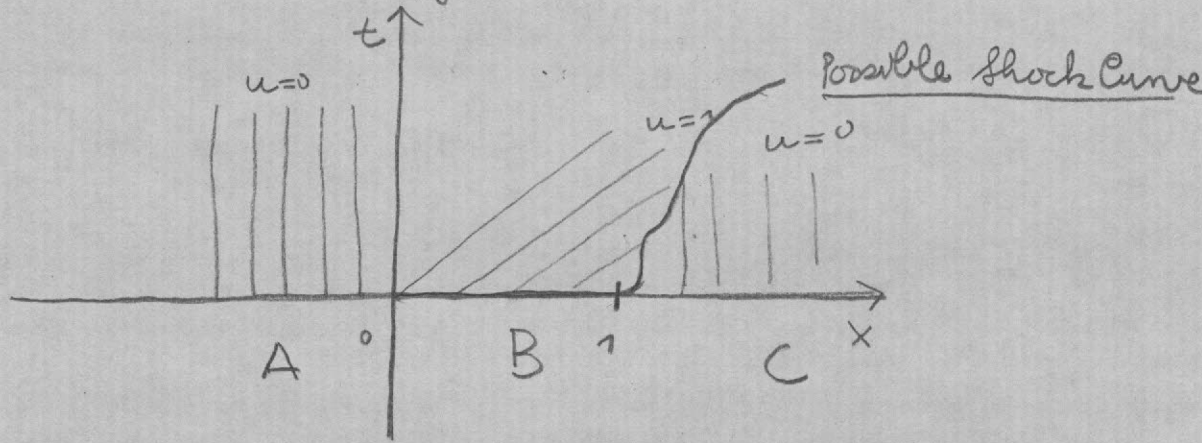
(10) a) We observe that $u_t + (q(u))_x = 0$ for $q(u) = \frac{1}{2}u^2$.
$q'(u) = u$.
We have the following characteristics:



b) We see that a shock forms where the characteristics from region B meet the characteristics from region C.

c) We observe that the shock curve $(s(t), t)$ starts from $(1,0)$, i.e. $s(0) = 1$.
Furthermore $u_- = 1$, $u_+ = 0$.
By the Rankine-Hugoniot Condition:

$$\dot{s} = \frac{q(u_+) - q(u_-)}{u_+ - u_-} = \frac{\frac{1}{2}\cdot 0^2 - \frac{1}{2}\cdot 1^2}{0-1} = \frac{1}{2}.$$

$$\Rightarrow \underline{s(t) = \frac{1}{2}\cdot t + 1}$$