18.152: Fall 2010
Practice final

These problems are examples of problems that may be given in the final. The length of the practice final may not represent the actual length of the final itself. Solutions will be posted during the weekend.

1. Let $D > 0$ be constant, $g \in C^1([0, \pi])$ and $g'(0) = g'(\pi) = 0$.

(a) Using separation of variables solve the initial-boundary value problem

$$\begin{cases} u_t - Du_{xx} = 0 & 0 < x < \pi, \ 0 < t \\ u(x, 0) = g(x) & 0 \leq x \leq \pi \\ u_x(0, t) = 0, \ u(\pi, t) = 1 & 0 < t. \end{cases}$$

(b) Prove that the solution is unique.

2. Consider the problem

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = x & 0 < x < 1, \ 0 < t \\ u(x, 0) = 0 & 0 \leq x \leq 1 \\ u(0, t) = u(1, t) = 0 & 0 < t. \end{cases} \tag{0.1}$$

(a) Find the stationary $(u_t = 0)$ solution $u^s(x) = u(x, t)$ for the equation and the boundary data in problem (0.1).

(b) Prove that $u(x, t) \leq u^s(x)$ for $t > 0$.

(c) Find a function $f(t) > 0$ such that

$$\lim_{t \to \infty} f(t) = 1 \quad \text{and} \quad u(x, t) \geq f(t) u^s(x)$$

(d) Deduce that $u(x, t) \to u^s(x)$ as $t \to \infty$ uniformly in $[0, 1]$.

3. Let B_R be a ball of radius R in $\mathbb{R}^3$ made of some homogeneous material with surface temperature $U > 0$ constant at time $t = 0$. Describe how the temperature of B_R evolves in each interior point if the surface of the ball is kept at zero temperature. Check that at the center the temperature tends to zero exponentially as $t \to \infty$.
Hint: Use the identity

$$u_t - \Delta u = u_t - (u_{rr} + \frac{2}{r} u_r) = u_t - \frac{1}{r}(ru)_{rr}$$

that holds for radial function and look for $v := ru$.

4. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Prove that the solution to the Dirichlet problem

$$\begin{cases} \Delta u(x) = f(x) & x \in \Omega \\ u(\sigma) = h(\sigma) & \sigma \in \partial\Omega \end{cases}$$

is given by

$$u(x) = -\int_{\partial\Omega} h(\sigma)\partial_\nu G(x,\sigma) - \int_\Omega f(y)G(x,y)dy,$$

where G is the Green function for Ω.

5. Let B_1 be the unit ball centered at the origin in $\mathbb{R}^3$. Let u be the solution to the Dirichlet problem

$$\begin{cases} \Delta u(x,y,z) = 0 & \text{in } B_1 \\ u(x,y,z) = x^4 + y^4 + z^4 & \text{on } \partial B_1. \end{cases}$$

Compute max and min of u on the closure of the ball B_1.

6. Let u be harmonic in $\mathbb{R}^3$ and such that

$$\int_{\mathbb{R}^3} |\nabla u(x)|^2 \, dx < \infty.$$

Prove that u is constant.

(*Hint:* Use the mean value property for ∇u).

7. Find the solution for the problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t) & 0 < x < L, \ 0 < t \\ u(x,0) = u_t(x,0) = 0 & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & 0 < t, \end{cases}$$

when

$$f(x,t) = e^{-t} \sin\left(\frac{\pi x}{L}\right) \quad \text{and when} \quad f(x,t) = xe^{-t}.$$

8. Show that the solution of the 2D non homogeneous Cauchy problem

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in } \mathbb{R}^2 \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

is

$$u(x,t) = \frac{1}{2\pi c} \int_0^t \int_{B_{c(t-s)}(x)} \frac{f(y,s)}{\sqrt{c^2(t-s)^2 - |x-y|^2}} dyds.$$

9. Consider the equation

$$u_{tt} - tu_{xx} = 0.$$

(a) Classify the equation.
(b) Find the associated characteristics.
(c) Write the equation in canonical form and find the general solution.

10. Consider the problem

$$\begin{cases} u_t + uu_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 0 & x < 0, \\ 1 & 0 < x < 1, \\ 0 & 1 < x. \end{cases}$$

(a) Write the solution $u(x,t)$ in terms of the function g by using the method of characteristics.
(b) By representing some of the characteristics what can you say about a possible shocks curve for the solution u?
(c) Can you find the equation of such a curve if it exists?