SOLUTIONS

18.152: Fall 2010
Midterm Exam

Each problem is worth 20 points. Partial credit is given only for statements clearly written and following logically.

[20 points] 1. Let $D > 0$. Use separation of variables to solve the Cauchy problem with mixed boundary data

$$\begin{cases} u_t(x,t) - Du_{xx}(x,t) = 0 & 0 < x < \pi,\ 0 < t \\ u(x,0) = 2\sin(\frac{3}{2}x) & 0 \le x \le \pi \\ u(0,t) = u_x(\pi,t) = 0 & 0 < t. \end{cases}$$

We suppose $\tilde{u}(x,t) = v(x) \cdot w(t)$ solves:

$$\begin{cases} \tilde{u}_t - D\tilde{u}_{xx} = 0 & ; 0 < x < \pi,\ 0 < t \\ \tilde{u}(0,t) = \tilde{u}_x(\pi,t) = 0 \end{cases}$$

$$\Rightarrow v(x) \cdot w'(t) - D v''(x) w(t) = 0$$

$$\Rightarrow \frac{v''(x)}{v(x)} = \frac{w'(t)}{D w(t)} = \text{const.} = \lambda = -\mu^2$$ [5 points]

$$v(x) = A\sin(\mu x) + B\cos(\mu x)$$

We also know: $v(0) = v_x(\pi) = 0$

$$\Rightarrow \mu_k = \frac{2k+1}{2} \text{ for } k \in \mathbb{N}_0,\ B = 0$$

$$v_k(x) = A_k \sin\left(\frac{2k+1}{2}\right)$$ [5 points]

$$\frac{w'(t)}{Dw(t)} = -\left(\frac{2k+1}{2}\right)^2;\quad w_k(t) = w_k(0) \cdot e^{-\left(\frac{2k+1}{2}\right)^2 Dt}$$

So: we want $u(x,t) = \sum_{k=0}^{\infty} C_k e^{-\left(\frac{2k+1}{2}\right)^2 Dt} \sin\left(\frac{2k+1}{2}x\right)$ [5 points]

$$u(x,0) = 2\sin\left(\tfrac{3}{2}x\right) \Rightarrow \boxed{\begin{array}{l} C_1 = 2 \\ C_k = 0 \text{ for } k \neq 1 \end{array}}$$

Therefore: $\boxed{u(x,t) = 2e^{-\frac{9}{4}Dt} \cdot \sin\left(\tfrac{3}{2}x\right)}$ [5 points]

[20 points] **2.** Let $u(x,t)$ be a solution to the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & 0 < x < 1,\ 0 < t \\ u(x,0) = \sin \pi x & 0 \le x \le 1 \\ u(0,t) = 2te^{(1-t)}, \quad u(1,t) = 1 - \cos \pi t & 0 < t. \end{cases}$$

(a) Prove that $u \ge 0$.

(b) Prove that $u\left(\frac{1}{2}, \frac{1}{8}\right) \le 1$ and $u\left(\frac{1}{2}, 3\right) \le 2$.

a) We note that:

$$\begin{cases} \sin \pi x \ge 0 & \forall\ 0 \le x \le 1 \\ 2te^{1-t} \geq 0 & \forall\ 0 < t \\ 1 - \cos \pi t \ge 0 & \forall\ 0 < t \end{cases}$$

By the Minimum Principle, it follows that $\boxed{u \ge 0}$ [8 points]

b) We observe:

$\sin \pi x \le 1 \quad \forall 0 \le x \le 1 \qquad (\mathrm{I})$

Let $f(t) := 2te^{1-t}$. Then $f'(t) = 2(1-t)e^{1-t}$

So f achieves its global maximum at 1, and $f(1) = 2$.

Also, f is increasing on $[0,1]$, so:

$$\begin{cases} f(t) \le f\left(\frac{1}{8}\right) = \frac{1}{4} e^{1-\frac{1}{8}} = \frac{1}{4} e^{\frac{7}{8}} < \frac{3}{4} < 1 \quad \forall\ 0 < t \le \frac{1}{8} \quad (\mathrm{II}a) \\ f(t) \le f(1) = 2 \quad \forall\ 0 < t \quad (\mathrm{II}b) \end{cases}$$

Finally

$$\begin{cases} 1 - \cos \pi t \le 1 \quad \forall\ 0 < t \le \frac{1}{2}. & (\mathrm{III}\,a) \\ 1 - \cos \pi t \le 2 \quad \forall 0 < t & (\mathrm{III}\,b) \end{cases}$$

Now, (I), $(\mathrm{II}a)$, $(\mathrm{III}a)$ and the Maximum Principle imply:

$\boxed{u\left(\frac{1}{2}, \frac{1}{8}\right) \le 1}$ [6 points]

Furthermore, (I), $(\mathrm{II}b)$, $(\mathrm{III}^2 b)$, and the Maximum Principle imply:

$\boxed{u\left(\frac{1}{2}, 3\right) \le 2}$. [6 points]

[20pts]

3. Consider the symmetric random walk on the line we discussed in class. Suppose that a perfectly reflecting barrier is located at the point $L = \bar{m}h + h/2$. By this we mean that if the particle hits the point $L - h/2$ at time t and moves to the right, then it is reflected and it comes back to $L - h/2$ at time $t + \tau$. Show that for $h, t \to 0$ and $h^2/\tau = 2D$, $p = p(x,t)$ is a solution to the problem

$$\begin{cases} p_t(x,t) - Dp_{xx}(x,t) = 0 & x < L,\ 0 < t \\ p(x,0) = \delta & 0 \le x \le L \\ p_x(L,t) = 0 & 0 < t \end{cases}$$

and moreover explain why it should be that $\int_{-\infty}^{L} p(x,t)\, dx = 1$.

We really only have to check the Neumann boundary condition $p_x(L,t) = 0$ since for $x < L$ the law for $p(t,x)$ is like the one in class.

We start by writing

$$p(L - \tfrac{h}{2}, t+\tau) = \underbrace{\tfrac{1}{2} p(L - \tfrac{h}{2}, t)}_{\text{reflected part}} + \underbrace{\tfrac{1}{2} p(L - h\tfrac{3}{2}, t)}_{\text{part coming from the left}}$$

Since $p(L - \frac{h}{2}, t+\tau) = p(L - \frac{h}{2}, t) + p_t(L - \frac{h}{2}, t^*)\tau + O(\tau^2)$

we can write

$$p(L - \tfrac{h}{2}, t) - \tfrac{1}{2} p(L - \tfrac{h}{2}, t) - \tfrac{1}{2} p(L - \tfrac{3}{2}h, t) + p_t(L - \tfrac{h}{2}, t^*)\tau + O(\tau^2) = 0$$

hence

$$\tfrac{1}{2}\left[p(L - \tfrac{h}{2}, t) - p(L - \tfrac{3}{2}h, t)\right] \cdot \tfrac{1}{h} = \left[-p_t(L - \tfrac{h}{2}, t^*)\tau + O(\tau^2)\right]\tfrac{1}{h}$$

$\downarrow h \to 0$ $\qquad\qquad$ $\downarrow h \to 0$ since $\frac{\tau}{h^2} = c$

$\frac{1}{2} p_x(L,t)$ $\qquad\qquad$ 0

$$p_x(L,t) \equiv 0$$

$\boxed{\int_{-\infty}^{L} p(x,t)\, dx = 1 \text{ since the particle remains on } x < L \text{ by reflection.}}$

→ sorry this was a typo!

[20 pts] 4. Let $u(x, \cancel{y})$ be an harmonic function in $\mathbb{R}^n$.

(a) Is u^2 an harmonic function in $\mathbb{R}^n$?

(b) Prove that if $F(z)$ is a C^2 function on $\mathbb{R}$ and it is convex, then $w = F(u)$ is subharmonic, that is $\Delta w \geq 0$.

(c) Is u^2 subharmonic?

[5 pts] a) We need to check that $\Delta u^2 = 0$

$\partial_{x_i} u^2 = 2u\,\partial_{x_i} u$

$\partial^2_{x_i x_i} u^2 = 2(\partial_{x_i} u)^2 + 2u\,\partial^2_{x_i x_i} u \Rightarrow \Delta u^2 = 2|Du|^2 + 2u\Delta u$, where $2|Du|^2 \geq 0$ and could be > 0, and $\Delta u = 0$.

So the answer is no in general.

[5 pts] c) this is already clear from a) since we proved $\Delta u^2 \geq 0$. But it comes also from the more general statement on (b) since the function $F(z) = z^2$ is convex.

[10 pts] b) $\partial_{x_i} F(u) = F'(u)\,\partial_{x_i} u$

$\partial^2_{x_i x_i} F(u) = F''(u)(\partial_{x_i} u)^2 + F'(u)\,\partial^2_{x_i x_i} u$

So $\Delta F(u) = F''(u)|Du|^2 + F'(u)\Delta u$, with $\Delta u = 0$

If F is convex then $F'' \geq 0 \Rightarrow \Delta F(u) \geq 0$

So $F(u)$ is subharmonic.

5. (a) Prove that

$$u(x,y) = \frac{1}{n^2}\sinh(ny)\cos(nx)$$

solves the Cauchy problem

$$\begin{cases} \Delta u(x,\cancel{t}\,y) = 0 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_y(x,0) = \frac{1}{n}\cos(nx) & x \in \mathbb{R}, \end{cases}$$

where $n \geq 1$.

(b) Deduce from here that the Cauchy problem for the Laplace operator Δ is not well posed, in the sense that there is no continuous dependence from the data.
(*Hint:* Note that by taking n large one can make $u_y(x,0)$ as small as one pleases. Is this also true for the corresponding solution u?)

[12pts] a) $\partial_x u(x,y) = +\frac{n}{n^2}\sin(nx)\sinh(ny)$

$\partial^2_{xx} u(x,y) = -\frac{n^2}{n^2}\cos(nx)\sinh(ny)$

$\partial^2_{yy} u(x,y) = \frac{n^2}{n^2}\cos(nx)\sinh(ny) \Rightarrow \Delta u \equiv 0$ ✓

$u(x,0) = \frac{1}{n^2}\sinh(0)\cos(nx) = 0$ ✓

$u_y\Big|_{y=0} = \frac{n}{n^2}\cosh(ny)\cos(nx)\Big|_{y=0} = \frac{1}{n}\cos(nx)$ ✓

[8pts] b) $\lim_{n\to+\infty} u(x,y) = \lim_{n\to+\infty} \frac{\overset{+\infty}{e^{ny}} - \overset{\to 0 \text{ since } y>0}{e^{-ny}}}{2n^2}\cos(nx)$

since $\lim_{n\to+\infty}\frac{e^{ny}}{2n^2} = +\infty$ for $y>0$ — this part oscillate between -1 and 1

it follows that actually the solution oscillates between $-\infty$ and $+\infty$!

$\lim_{n\to\infty}\frac{1}{n}\cos(nx) = 0$ this shows that given $\varepsilon > 0$ we can take n_1, n_2 large s.t. $\left|\frac{1}{n_1}\cos(n_1 x) - \frac{1}{n_2}\cos(n_2 x)\right| < \varepsilon$ but the solutions are not close!