Solutions

18.152: Fall 2010
Final Exam

Please write only what you want us to grade. The maximum grade you could receive is 100 points.

1. [15pts] Suppose Ω is a smooth bounded domain, g is a smooth function on $\overline{\Omega}$ and u is a smooth solution of

$$\begin{cases} u_t - \Delta u + \gamma u = 0 & \text{in } x \in \Omega,\ t > 0 \\ u(\sigma, t) = 0 & \sigma \in \partial\Omega,\ t > 0 \\ u(x,0) = g(x) & x \in \Omega, \end{cases}$$

with $\gamma > 0$ constant. Prove that

$$|u(x,t)| \le Ce^{-\gamma t}$$

for all $(x,t) \in \Omega \times [0,T]$, and $T > 0$.

Solution: Let $v(x,t) := u(x,t)e^{\gamma t}$. Then, one has:

$$v_t(x,t) = u_t(x,t)e^{\gamma t} + \gamma u(x,t)e^{\gamma t}$$

$$\Delta v(x,t) = \Delta u(x,t)e^{\gamma t}$$

Hence: $v_t - \Delta v = (u_t - \Delta u + \gamma u)e^{\gamma t} = 0$

Also: $v(\sigma,t) = 0$; $\sigma \in \partial\Omega, t>0$

$v(x,0) = g(x)$.

Hence, v solves:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } x \in \Omega,\ t>0 \\ v(\sigma,t) = 0 & \sigma \in \partial\Omega,\ t>0 \\ v(x,0) = g(x) & x \in \Omega \end{cases}$$

So, by the Maximum Principle $v(x,t) \le \sup_{x\in\Omega} g(x) \le C_1 < \infty$ by the fact that g is smooth on $\overline{\Omega}$, and Ω is compact.

$\Rightarrow$ $u(x,t) \le C_1 e^{-\gamma t}$ (1). Similarly $(-v)$ solves:

$\forall (x,t) \in \Omega \times [0,T]\ \forall T>0$

$$\begin{cases} (-v)_t - \Delta(-v) = 0, & x \in \Omega\ t> \\ (-v)(\sigma,t) = 0, & \sigma \in \ \ t \\ (-v)(x,0) = (-g)(x) \end{cases}$$

and $(-v)(x,t) \le \sup_{x\in\Omega}(-g(x)) \le C_2 < \infty$ and hence:

$u(x,t) \ge -C_2 e^{-\gamma t}$ $\forall (x,t) \in \Omega \times [0,T]\ \forall T>0$ (2). Let $C := \max\{C_1, C_2\} > 0$.

(1) and (2) imply the claim. $\square$

2. [10pts] Using Fourier transform with respect to x, find the formula for the solution of the Cauchy problem

$$\begin{cases} u_t - D\Delta u = f(x,t) & x \in \mathbb{R}^n,\ t>0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n, \end{cases}$$

where $D > 0$, $g \in L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n \times \mathbb{R})$.
Hint: You will need to know that anti Fourier transform of $e^{-D|\xi|^2 t}$ is

$$\Gamma_D(x,t) = \frac{1}{(4\pi D t)^{n/2}} e^{-\left(\frac{|x|^2}{4Dt}\right)}.$$

Solution: We write $u = v + w$ where v and w solve:

$$(*) \begin{cases} v_t - D\Delta v = 0 \\ v(x,0) = g(x) \end{cases} \quad \text{and } (**) \begin{cases} w_t - D\Delta w = f(x,t) \\ w(x,0) = 0 \end{cases}$$

We first find v. Taking Fourier transforms in x, it follows that:

$$\begin{cases} \hat{v}_t + D|\xi|^2 \hat{v} = 0 \\ \hat{v}(\xi, 0) = g(\xi) \end{cases}$$

$$\Rightarrow \hat{v}(\xi,t) = e^{-D|\xi|^2 t}\hat{g}(\xi)$$

$$\Rightarrow v(x,t) = \left(\left(e^{-D|\xi|^2 t}\right)^{\vee} * g\right)(x,t) = (\Gamma_D * g)(x,t) =$$

$$= \int_{\mathbb{R}^n} \Gamma_D(x-y,t) g(y)\,dy = \frac{1}{(4\pi D t)^{\frac{n}{2}}} \cdot \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4Dt}} \cdot g(y)\,dy \qquad (1)$$

On the other hand, by Duhamel's Principle:

$$w(x,t) = \int_0^s \tilde{w}(x,s;s)\,ds, \text{ where}$$

$$\begin{cases} \tilde{w}_t(\cdot,\cdot;s) - D\Delta \tilde{w}(\cdot,\cdot;s) = 0\ ;\ x \in \mathbb{R}^n, t>0 \\ \tilde{w}(x,s;s) = f(x,s) \end{cases}$$

Hence, by using the calculations ~~to~~ obtain (1), it follows that:

$$\tilde{w}(x,t;s) = \int_{\mathbb{R}^n} \Gamma_D(x-y,t-s) f(y,s)\,dy\,ds$$

$$\Rightarrow w(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi D(t-s))^{\frac{n}{2}}} \cdot e^{-\frac{|x-y|^2}{4D(t-s)}} \cdot f(y,s)\,dy\,ds \qquad (2)$$

(1), (2) then imply:

$$u(x,t) = \frac{1}{(4\pi D t)^{\frac{n}{2}}} \cdot \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4Dt}} \cdot g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi D(t-s))^{\frac{n}{2}}} \cdot e^{-\frac{|x-y|^2}{4D(t-s)}} \cdot f(y,s)\,dy\,ds. \quad \square$$

3. [15pts]

[5pts] (a) Suppose that $u(x,t)$ is smooth and solves the heat equation

$$u_t - D\Delta u = 0 \quad x \in \mathbb{R}^n,\ t > 0.$$

Let $\lambda > 0$. Show that

$$u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$$

is also solution of the same equation for all $\lambda \in \mathbb{R}$.

[10pts] (b) Use (a) to show that

$$v(x,t) := x\nabla u(x,t) + 2tu_t(x,t)$$

also solves the heat equation.

Solution: a) By the Chain Rule:

$$\frac{\partial}{\partial t}(u_\lambda(x,t)) = \lambda^2 \cdot \left(\frac{\partial}{\partial t} u\right)(\lambda x, \lambda^2 t)$$

$$\frac{\partial}{\partial x_j}(u_\lambda(x,t)) = \lambda \cdot \left(\frac{\partial}{\partial x_j} u\right)(\lambda x, \lambda^2 t)$$

$$\frac{\partial^2}{\partial x_j^2}(u_\lambda(x,t)) = \lambda^2 \cdot \left(\frac{\partial^2}{\partial x_j^2} u\right)(\lambda x, \lambda^2 t)$$

$$\Rightarrow \Delta u_\lambda(x,t) = \lambda^2 (\Delta u)(\lambda x, \lambda^2 t)$$

Hence: $\left(\frac{\partial}{\partial t} u_\lambda - D\Delta u_\lambda\right)(x,t) = \lambda^2\left(\frac{\partial}{\partial t} u - D\Delta u\right)(\lambda x, \lambda^2 t) = 0$

b) From a), we have:

$$\frac{\partial}{\partial t} u_\lambda - D\Delta u_\lambda = 0$$

We then take $\frac{\partial}{\partial \lambda}$ on both sides of the equation to deduce:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda} u_\lambda\right) - D\Delta\left(\frac{\partial}{\partial \lambda} u_\lambda\right) = 0 \quad \forall \lambda > 0$$

(Here, we are using the fact that derivatives in λ commute with derivatives in t and x_j.)

Furthermore, by the Chain Rule:

$$\frac{\partial}{\partial \lambda} u_\lambda(x,t) = x \cdot \nabla u(\lambda x, \lambda^2 t) + 2t\lambda\, u_t(\lambda x, \lambda^2 t)$$

In particular:

$$\left(\frac{\partial}{\partial \lambda} u_\lambda(x,t)\right)\Big|_{\lambda=1} = x \cdot \nabla u(x,t) + 2t\, u_t(x,t) = v(x,t).$$

Hence, v also solves the heat equation. $\square$

4. [15pts] Use separation of variables to solve in the rectangle $Q = \{(x,y) : 0 < x < a,\ 0 < y < b\}$ the problem

$$\begin{cases} \Delta u = 0 & \text{in } Q \\ u(x,0) = 0,\ u(x,b) = g(x) & 0 \le x \le a \\ u(0,y) = u_x(a,y) = 0 & 0 \le y \le b, \end{cases}$$

where $g \in C^1(\mathbb{R})$, and $g(0) = g'(a) = 0$.

Solution: We observe that the boundary conditions at $x=0$, $x=a$ are satisfied by:

$$w_k(x,y) := \sin\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)\cdot x\right)\cdot f_k(y)$$

w_k solves the Laplace equation if:

$$-\frac{\pi^2}{a^2}\left(k+\frac{1}{2}\right)^2 \sin\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)\cdot x\right)\cdot f_k(y) + \sin\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)\cdot x\right)\cdot f_k''(y) = 0$$

$$\Rightarrow f_k''(y) = \frac{\pi^2}{a^2}\cdot\left(k+\frac{1}{2}\right) f_k(y) \Rightarrow f_k(y) = a_k e^{\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)y} + b_k e^{-\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)y}$$

So, we are looking for:

$$u(x,y) = \sum_{k=0}^{+\infty}\left\{a_k \sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)\cdot e^{\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)y} + b_k \sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)\cdot e^{-\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)y}\right\}$$

· Let us consider $y=0$.

$$\Rightarrow \sum_{k=0}^{+\infty}\left(a_k+b_k\right)\cdot\sin\left(\frac{\pi\left(k+\frac{1}{2}\right)}{a}\cdot x\right) = 0 \quad \forall\, 0 \le x \le a$$

It follows that $a_k + b_k = 0 \quad \forall\, k = 0, 1, \ldots$

Hence, letting $c_k := 2a_k$, we obtain $u(x,y) = \sum_{k=0}^{+\infty} c_k\cdot\sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)\cdot\sinh\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)y\right)$.

We want: $u(x,b) = g(x)$, i.e.,

$$\sum_{k=0}^{+\infty} c_k\cdot\sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)\cdot\sinh\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)b\right) = g(x), \ \forall\, 0 \le x \le a$$

We write $g(x) = \sum_{k=0}^{+\infty} g_k \sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)$ (such an expansion exists by the assumptions on g).

Then $\boxed{g_k = \frac{2}{a}\int_0^a g(x)\cdot\sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)dx}$,

Now, $\boxed{c_k = \frac{1}{\sinh\left(\frac{\pi}{a}\cdot\left(k+\frac{1}{2}\right)b\right)}\cdot g_k}$

For c_k as constructed, we have:

$$\boxed{u(x,y) = \sum_{k=0}^{+\infty} c_k\cdot\sin\left(\frac{\pi\cdot\left(k+\frac{1}{2}\right)}{a}x\right)\cdot\sinh\left(\frac{\pi\left(k+\frac{1}{2}\right)}{a}y\right)}$$

□

5. [15] Discuss the well-posedness of the problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in the set } \{(x,t) : x > t\} \\ u(x,x) = f(x) & x \in \mathbb{R} \\ \partial_\nu u(x,x) = g(x) & x \in \mathbb{R}, \end{cases}$$

where $\nu = \frac{1}{\sqrt{2}}(1,-1)$, in terms of the function g.

Hint: First write the generic solution $u(x,t) = F(x-t) + G(x+t)$ and then check when the conditions are compatible.

Solution: We first write $u(x,t) = F(x-t) + G(x+t)$. It follows that:

$\partial_{\vec{\nu}} u(x,x) = \vec{\nu} \cdot \nabla u \ (x,x)$, where:

$\nabla u(x,t) = \langle F'(x-t) + G'(x+t), -F'(x-t) + G'(x+t) \rangle$

So: $\vec{\nu} \cdot \nabla u(x,x) = \frac{1}{\sqrt{2}} (F'(0) + G'(2x)) - \frac{1}{\sqrt{2}} (-F'(0) + G'(2x)) = \underline{\sqrt{2} F'(0) = k = \text{const.}}$ (1)

Hence, in order for the PDE to have any solution it is necessary that $\underline{g(x) \equiv k \text{ is constant.}}$

Suppose now that $g(x) \equiv k$.

$u(x,x) = F(0) + G(2x) = f(x) \Rightarrow \boxed{G(x) = f\left(\frac{x}{2}\right) - F(0)}$ (2)

Hence $\underline{u(x,t) = F(x-t) + f\left(\frac{x+t}{2}\right) - F(0)}$ is a solution of the problem if F satisfies (1). Clearly, $F(x) = \frac{1}{\sqrt{2}} \cdot k \cdot x$ satisfies (1), so $u(x,t) = \frac{1}{\sqrt{2}} k (x-t) + f\left(\frac{x+t}{2}\right)$ is an explicit solution.

However, for any $r \in \mathbb{R}$, $F_r(x) = \frac{1}{\sqrt{2}} k \cdot x + rx^2$ also satisfies (1), and for $r \neq 0$ gives us a different solution:

$$\underline{u^{(r)}(x,t) = \frac{1}{\sqrt{2}} k (x-t) + r(x-t)^2 + f\left(\frac{x+t}{2}\right)}$$

In fact, in this way, we obtain infinitely many different solutions to the same problem. Hence, <u>this problem is not well-posed.</u> □

6. [20pts] Consider the linear, second order equation

$$2u_{xx} + 6u_{xy} + 4u_{yy} + u_x + u_y = 0.$$

[5pts] (a) Classify the equation and find the characteristics.

[15pts] (b) Write the equation in canonical form and determine the general solution.

Solution: a) We compute: $D = 3^2 - 4 \cdot 2 = 9 - 8 = 1 > 0$, hence the equation is hyperbolic.

$2t^2 + 6t + 4 = 2(t+2)(t+1)$, so we can factorize the principal part as:

$2u_{xx} + 6u_{xy} + 4u_{yy} = 2(\partial_x + 2\partial_y)(\partial_x + \partial_y)u$

$(\partial_x + 2\partial_y)\varphi = 0$ is satisfied for $\varphi(x,y) = 2x - y$

$(\partial_x + \partial_y)\psi = 0$ is satisfied for $\psi(x,y) = x - y$

Hence, the characteristics are:

$2x - y = c_1, \; x - y = c_2.$

b) Let: $\boxed{\xi := 2x - y, \; \eta := x - y.}$

Then: $x = \xi - \eta, \; y = \xi - 2\eta$

Suppose $u(x,y) = \mathcal{U}(\xi(x,y), \eta(x,y))$. We then obtain:

$u_x = 2\mathcal{U}_\xi + \mathcal{U}_\eta$

$u_y = -\mathcal{U}_\xi - \mathcal{U}_\eta$

$u_{xx} = 4\mathcal{U}_{\xi\xi} + 4\mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\eta}$

$u_{xy} = -2\mathcal{U}_{\xi\xi} - 3\mathcal{U}_{\xi\eta} - \mathcal{U}_{\eta\eta}$

$u_{yy} = \mathcal{U}_{\xi\xi} + 2\mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\eta}$

$\Rightarrow 8\mathcal{U}_{\xi\xi} + 8\mathcal{U}_{\xi\eta} + 2\mathcal{U}_{\eta\eta} - 12\mathcal{U}_{\xi\xi} - 18\mathcal{U}_{\xi\eta} - 6\mathcal{U}_{\eta\eta} + 4\mathcal{U}_{\xi\xi} + 8\mathcal{U}_{\xi\eta} + 4\mathcal{U}_{\eta\eta}$
$+ 2\mathcal{U}_\xi + \mathcal{U}_\eta - \mathcal{U}_\xi - \mathcal{U}_\eta = 0$

$\boxed{-2\mathcal{U}_{\xi\eta} + \mathcal{U}_\xi = 0}$; $\mathcal{U}_{\xi\eta} = \frac{1}{2}\mathcal{U}_\xi \Rightarrow \mathcal{U}_\xi = e^{\frac{1}{2}\eta} \cdot f(\xi)$; Let $F' = f$

Canonical Form

$\Rightarrow \mathcal{U} = e^{\frac{1}{2}\eta} \cdot F(\xi) + G(\eta)$

So: $\boxed{u(x,y) = e^{\frac{x-y}{2}} F(2x-y) + G(x-y)}$. $\square$

General Solution

7. [10pts] Consider the non homogeneous problem

$$\begin{cases} c_t + vc_x = f(x,t) & x \in \mathbb{R},\ t>0 \\ c(x,0) = 0 & x \in \mathbb{R}, \end{cases}$$

where v is a nonzero constant.

[5ptos] (a) Using the Duhamel's principle write the solution for this problem.

[5ptos] (b) Find an explicit formula when $f(x,t) = e^{-t}\sin x$.

Solution: a) Let $0 \le s \le t$. We look at the problem:

$$\begin{cases} w_t(x,t;s) + v\,w_x(x,t;s) = 0 & ;\ x\in\mathbb{R},\ t>0. \\ w(x,s;s) = f(x,s) & ;\ x \in \mathbb{R} \end{cases}$$

We can use the method of characteristics:

$$w(x,t;s) = f(x-v(t-s),s)$$

Hence, by Duhamel's Principle:

$$c(x,t) = \int_0^t w(x,t;s)\,ds = \int_0^t f(x-v(t-s),s)\,ds$$

b) When $f(x,t)=e^{-t}\sin x$, it follows that:

$$c(x,t) = \int_0^t e^{-s}\sin(x-v(t-s))\,ds = \operatorname{Im}\left\{\int_0^t e^{-s} e^{i(x-v(t-s))}\,ds\right\} =$$

$$= \operatorname{Im}\left\{e^{i(x-vt)} \cdot \int_0^t e^{s(-1+iv)}\,ds\right\} = \operatorname{Im}\left\{e^{i(x-vt)} \cdot \frac{1}{-1+iv}\left(e^{t(-1+iv)}-1\right)\right\}$$

$$= \operatorname{Im}\left\{\frac{1}{-1+iv}\cdot e^{ix}\cdot e^{-t} - \frac{1}{-1+iv}e^{i(x-vt)}\right\}$$

$$= \operatorname{Im}\left\{\frac{-1-iv}{1+v^2}\cdot(\cos x + i\sin x)\,e^{-t} - \frac{-1-iv}{1+v^2}\left(\cos(x-vt)+i\sin(x-vt)\right)\right\} =$$

$$= -\frac{e^{-t}}{1+v^2}\cdot\sin x - \frac{ve^{-t}}{1+v^2}\cdot\cos x + \frac{\sin(x-vt)}{1+v^2} + \frac{v\cos(x-vt)}{1+v^2}. \quad \square$$