Solution to Practice Problems

Problem # 1: Part(a)

$$v_t = u_t e^{hx+tk} + ku\, e^{hx+tk}$$

$$v_x = u_x e^{hx+tk} + u h e^{hx+tk}$$

$$v_{xx} = u_{xx} e^{hx+tk} + h 2 u_x e^{hx+tk} + u h^2 e^{hx+tk}$$

$$v_t = D v_{xx} \iff e^{hx+tk} [u_t + ku = D u_{xx} + 2h D u_x + + D h^2 u]$$

$$u_t = D u_{xx} + 2h D u_x + [D h^2 - k] u$$

set $2hD = b \quad Dh^2 - k = c$

$$h = \frac{b}{2D} \qquad k = Dh^2 + c$$

Part(b)

Using part(a) if we set $w(x,t) = u(x,t) e^{-mt}$
we can solve

$$\begin{cases} w_t - w_{xx} = e^{-mt} [\sin 2\pi x + 2 \sin 3\pi x] \\ w(x,0) = 0 \\ w(0,t) = 0 = w(1,t) \end{cases}$$

We use separation of variables: $w(x,t) = V(x) Z(t)$

Boundary data are here!

The eigenfunctions are

$$v_k(x) = \sin k\pi x \qquad k = 1, 2, \ldots$$

and eigenvalues $\lambda_k = -k^2\pi^2$

Since the non-homogeneous part of the equation is of the form $e^{-mt}[v_2(x) + 2v_3(x)]$ we are looking for a solution of the form:

$$w(x,t) = c_2(t)\sin 2\pi x + 2c_3(t)\sin 3\pi x$$

Since

$$w_t - w_{xx} = [c_2'(t) + 4\pi^2 c_2(t)]\sin 2\pi x + 2[c_3'(t) + 9\pi^2 c_3(t)]\sin 3\pi x$$

we have to set

$$c_2'(t) + 4\pi^2 c_2(t) = e^{-mt}$$

$$c_3'(t) + 9\pi^2 c_3(t) = e^{-mt}$$

So if $m \neq j^2\pi^2 \quad j = 2, 3$

$$c_j(t) = \frac{1}{j^2\pi^2 - m}\left(e^{-mt} - e^{-j^2\pi^2 t}\right)$$

$$m = j^2\pi^2 \qquad c_j(t) = t e^{-j^2\pi^2 t}$$

Since from the zero Dirichlet data we also have

$$c_j(0) = 0.$$

then $u(x,t) = e^{mt}[c_2(t)\sin 2\pi x + 2c_3(t)\sin 3\pi x]$

Problem #2 Part (a)

We are looking for the solution to

$$\begin{cases} -\psi''(x) = 1 \\ \psi(0) = \psi(1) = 0 \end{cases}$$

By integration we obtain

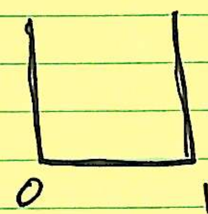
$$u^s(x) = \frac{1}{2} x(1-x)$$

Part (b)

Set $v(x,t) = u^s(x) - u(x,t)$. then

$$v_t - v_{xx} = 0 \qquad 0 < x < 1, \; t > 0$$

By the max principle it is enough to check what happens on the boundary:



$$v(x,0) = u^s(x) - u(x,0) = \frac{1}{2} x(1-x) \geq 0$$

$$v(1,t) = u^s(1) - u(1,t) = 0 - 0 = 0$$

$$v(0,t) = u^s(0) - u(0,t) = 0 - 0 = 0$$

So we conclude $v(x,t) \geq 0 \quad 0 \leq x \leq 1 \quad t \geq 0$

$$\Rightarrow \quad u^s(x) \geq u(x,t) \quad \text{for } t > 0$$

Part (c)

Set $w(x,t) = u(x,t) - (1 - e^{-\beta t}) u^s(x)$

$$\partial_t \left[(1-e^{-\beta t})u^s(x)\right] = -\frac{\beta}{2}x(1-x)e^{-\beta t}$$

$$\partial_{xx}\left[\quad '' \quad\right] = -e^{-\beta t}$$

So we obtain

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) = 1 + e^{-\beta t}\left(1 - \frac{\beta}{2}x + \frac{\beta}{2}x^2\right) & \\ w(x,0) = 0 & 0 \le x \le 1 \\ w(0,t) = w(1,t) = 0 & t > 0 \end{cases}$$

Since at the boundary we have zero values for w we want to have in $0 \le x \le 1$ and $t > 0$

$$1 + e^{-\beta t}\left(1 - \beta/_2\, x + \beta/_2\, x^2\right) \ge 0$$

So that we can use the max principle.

For example $\quad 1 - \beta/_2\, x + \beta/_2\, x^2 \ge 0 \quad 0 \le x \le 1$

$$\beta/_2\, x^2 - \beta/_2\, x + 1 = 0 \iff$$

if $\beta > 0 \quad x^2 - x + \frac{2}{\beta} = 0 \quad x = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{8}{\beta}}\right]$

so in particular if we don't have solutions the inequality is always satisfied: $1 - \frac{8}{\beta} < 0$

$\iff \frac{8}{\beta} > 1 \implies \beta < 8$ so $0 < \beta < 8$ is ok.

Then we conclude that $w \ge 0 \implies$

$$u(x,t) \ge (1 - e^{-\beta t})\, u^s(x).$$

Problem 3 : Part (a)

The Equation is of diffusion type hence if $u(x,t)$ is the density of a gas it is clear that if no reaction or drift is present the gas will tend to an equilibrium state that is uniformly the same at each point along the segment $[0,L]$.

Integrate the equation

$$\int_0^L u_t(x,t)\,dx = \int_0^L D\,u_{xx}(x,t)\,dx$$

~~Int by parts~~ Integrate

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = D\left[u_x(L,t) - u_x(0,t)\right] = 0$$

So $\frac{d}{dt}\int_0^L u(x,t)\,dx = 0$

hence $\int_0^L u(x,t)\,dx = C$ constant $= \int_0^L u(x,0)\,dx = \int_0^L g(x)\,dx$

$t \to +\infty$ $\Rightarrow$

$$\int_0^L U\,dx = UL = \int_0^L g(x)\,dx$$

$\Downarrow$

$$U = \frac{1}{L}\int_0^L g(x)\,dx.$$

Part (b): $E(t) = \int_0^L (u(x,t) - U)^2\,dx$

$$E'(t) = \int_0^L 2(u(x,t) - U)\cdot u_t(x,t)\,dx$$

(Note that the assumptions on u allow us to take derivative inside the integral !)

Now replace the value of u_t from equation

$$E'(t) = \int_0^L 2(u(x,t) - U)\, D u_{xx}(x,t)\, dx$$

Now integrate by parts

$$= -2D \int_0^L (u(x,t) - U)_x\, u_x\, dx =$$

$$= -2D \int_0^L (u(x,t) - U)_x^2\, dx = -2D \int_0^L w_x^2\, dx$$

<u>Claim</u>:

$$\int_0^L w_x^2\, dx \geq \frac{E(t)}{L^2}$$

to see this we write

$$|w(x,t)| = \left| \int_x^x w_x(s,t)\, ds \right| \leq \int_0^L |w_x(s,t)|\, ds$$

Fundamental theorem of calculus

$$\leq \left(\int_0^L w_x^2(s,t)\, ds \right)^{\frac{1}{2}} \left(\int_0^L 1^2\, ds \right)^{\frac{1}{2}} = \sqrt{L} \left(\int_0^L w_x^2(s,t)\, ds \right)^{\frac{1}{2}}$$

Cauchy - Schwartz inequality

So by squaring:

$$L\int_0^L w_x^2(\xi,t)\,d\xi \geq (w(x,t))^2$$

This for any arbitrary x, then if we integrate both sides in x we obtain

$$\underbrace{\int_0^L w^2(x,t)\,dx}_{= E(t)} \leq L^2 \int_0^L w_x^2(x,t)\,dx$$

and this proves the claim.

Using the claim and what we found for $E'(t)$ we get

$$E'(t) = -2D\int_0^L w_x^2\,dx \leq -\frac{2D}{L^2}E(t)$$

Since $E(t) > 0$ (unless $u \equiv U$) we have

$$\frac{E'(t)}{E(t)} \leq -\frac{2D}{L^2} \iff \frac{d}{dt}\left[\ln E(t)\right] \leq \frac{-2D}{L^2}$$

So by integration $\ln \frac{E(t)}{E(0)} \leq \frac{-2D}{L^2}t$

and by taking exp $\Rightarrow E(t) \leq E(0)\, e^{-\frac{2D}{L^2}t}$

and since $\frac{2D}{L^2} > 0 \Rightarrow$

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$$0 \le E(t) \le E(0)\, e^{-\frac{2D}{l^2}t}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \; t \to +\infty \\ 0 & & 0 \end{array}$$

hence $\lim_{t \to +\infty} E(t) \equiv 0$.

<u>Problem#4</u>: (a) is a consequence of the Green's theorem. (b) is obtained from (a) using $v = 1$.

<u>Problem #5</u>: Using Fourier Transform

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

and $\sum_{n=1}^{\infty}\left(|a_n| + |b_n|\right) < \infty$, $\sum_{n=1}^{\infty} |A_n| + |B_n| < \infty$

$C_{1,R}$

Due to the symmetry of the domain we will be using polar coordinates and separation of variables.

$$u(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty} [\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta]$$

the boundary conditions tell us that

$$\alpha_0(1) = \frac{a_0}{2} \qquad \alpha_0(R) = \frac{A_0}{2}$$

$$\alpha_n(1) = a_n \qquad \alpha_n(R) = A_n$$

$$\beta_n(1) = b_n \qquad \beta_n(R) = B_n$$

Since in polar coordinates

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad (\alpha)$$

it follows that

$$u_r(r,\theta) = \alpha_0'(r) + \sum_{n=1}^{\infty} \alpha_n'(r)\cos n\theta + \beta_n'(r)\sin n\theta$$

$$u_{rr}(r,\theta) = \alpha_0''(r) + \sum_{n=1}^{\infty} \alpha_n''(r)\cos n\theta + \beta_n''(r)\sin n\theta$$

$$u_{\theta\theta}(r,\theta) = -\sum_{n=1}^{\infty} n^2 [\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta]$$

Substituting in the equation (α) and matching using the 'n's :

① $\begin{cases} \alpha_0''(r) + \frac{1}{r}\alpha_0'(r) = 0 \\ \alpha_0(1) = \frac{a_0}{2} \quad \alpha_0(R) = \frac{A_0}{2} \end{cases}$

② $\begin{cases} \alpha_n''(r) + \frac{1}{r}\alpha_n'(r) - \frac{n^2}{r^2}\alpha_n(r) = 0 \\ \alpha_n(1) = a_n \quad \alpha_n(R) = A_n \end{cases}$

③ ... similarly for $\beta_n(r)$.

It is easy to see that a general solution is $C_1 + C_2 \log r$

$$\alpha_0(r) = \frac{a_0}{2} + \frac{A_0 - a_0}{2 \log R} \log r$$

found by imposing the conditions at $r=1$ and $r=R$

For ② and ③ we look for solutions of type

r^{γ} with γ to be determined

But imposing that this is solution we get

$$\left[\gamma(\gamma-1) + \gamma - n^2\right] r^{\gamma-2} = 0$$

from which $\gamma = \pm n$. So the general solution is of type $C_1 r^n + C_2 r^{-n}$ and by imposing the conditions at $r=1$ and $r=R$ one gets

$$\alpha_n(r) = a_n K_n(r)\, r^{-n} + A_n H_n(r) \left(\frac{r}{R}\right)^n$$

$$\beta_n(r) = b_n K_n(r)\, r^{-n} + B_n H_n(r) \left(\frac{r}{R}\right)^n$$

where $H_n(r) = \dfrac{1 - r^{-2n}}{1 - R^{-n}}$ and $K_n(r) = \dfrac{1 - R^{-2n} r^{2n}}{1 - R^{-2n}}$

Part (b): Since

for $g(\theta) \Rightarrow a_0 = 0,\ a_n = 0,\ \underset{n \neq 1}{b_n} = 0,\ b_1 = 1$

$h(\theta) \Rightarrow A_0 = 2,\ A_n = 0\ \ B_n = 0\ \ n \geq 1$

one finds $u(r,\theta) = \dfrac{\log r}{\log R} + \dfrac{R^2 - r^2}{(R^2-1) r} \sin\theta.$