

and bounded. We have

$$\begin{aligned}\|f - f_n\|_2 &= \left(\int_{(-1,1)} |f - f_n|^2 d\mu \right)^{1/2} \\ &\leq \left(\int_{[-1/n, 1/n]} 1^2 d\mu \right)^{1/2} \\ &\quad (\text{since } f \equiv f_n \text{ outside of } [-1/n, 1/n], \text{ and } |f - f_n| \leq 1 \text{ everywhere}) = \sqrt{2/n} \rightarrow 0,\end{aligned}$$

as $n \rightarrow \infty$. Thus f is in the L^2 -closure of M , but not in M itself, so M is not closed in L^2 .

20/20 ✓ **9.3** Assuming $M \neq H$, there exists some (nonzero) $w \in H \setminus M$. By the orthogonal projection theorem we can write $w = y + z$, where $y \in M, z \in M^\perp$. Since $w \notin M$ we have $z \neq 0$, and so $\text{span}(z) \in M^\perp$. In particular, $M^\perp$ has dimension at least 1. Now say $z, z' \in M^\perp$ are nonzero, and let $\alpha := L(z), \beta := L(z')$. By definition of M , $\alpha = 0$ iff $z \in M \cap M^\perp$, which is equivalent to $z = 0$, which we've assumed is not the case. So $\alpha \neq 0$, and similarly $\beta \neq 0$. Then we have

$$\begin{aligned}L\left(z - \frac{\alpha z'}{\beta}\right) &= L(z) - \frac{\alpha}{\beta} L(z') \\ &\quad (\text{by linearity}) \\ &= \alpha - \frac{\alpha}{\beta} \beta = 0,\end{aligned}$$

so that $w := z - \frac{\alpha z'}{\beta} \in M$. But also $w \in M^\perp$, since w is a linear combination of $z, z' \in M^\perp$. So $w \in M \cap M^\perp$, which means $w = 0$. Thus $z = \frac{\alpha z'}{\beta} \in \text{span}(z')$. It follows that $M = \text{span}(z')$, a space of dimension 1.