

$\frac{1}{2}(1+1) = 1 < \infty$. So $\frac{df}{dx} \in L^2(\mathbb{R})$. Also, $\|f\|_{L^2(\mathbb{R})}^2 = \int_{(-\infty,0)} e^{2x} d\mu + \int_{(0,\infty)} e^{-2x} d\mu < \infty$. So $f \in H$.

I couldn't show H is complete, but will give some starting observations. ✓

Suppose $\{f_n\} \subset H$ is a Cauchy sequence w.r.t. the metric induced by $\langle \cdot, \cdot \rangle_H$. Then $\frac{df_n}{dx}$ is defined μ -a.e. and is in $L^2(\mathbb{R})$, for each n . We have

$$\begin{aligned} \left\| \frac{df_n}{dx} - \frac{df_{n'}}{dx} \right\|_2^2 &= \left\| \frac{d}{dx}(f_n - f_{n'}) \right\|_2^2 \\ &= \left\langle \frac{d}{dx}(f_n - f_{n'}), \frac{d}{dx}(f_n - f_{n'}) \right\rangle_{L^2} \\ &\leq \left\langle \frac{d}{dx}(f_n - f_{n'}), \frac{d}{dx}(f_n - f_{n'}) \right\rangle_{L^2} + \langle (f_n - f_{n'}), (f_n - f_{n'}) \rangle_{L^2} \\ &= \langle f_n - f_{n'}, f_n - f_{n'} \rangle_H \\ &= \|f_n - f_{n'}\|_H^2 \rightarrow 0, \end{aligned}$$

as $n, n' \rightarrow \infty$. Thus $\{\frac{df_n}{dx}\}$ is a Cauchy sequence in $L^2(\mathbb{R})$. Similarly, $\{f_n\}$ is also L^2 -Cauchy.

Since $L^2(\mathbb{R})$ is complete, there exist f, g such that $\{f_n\} \rightarrow f$ in L^2 , and $\{\frac{df_n}{dx}\} \rightarrow g$ in L^2 . Let $D(f, m)(x) := \frac{f(x+1/m) - f(x)}{1/m}$, so that $\frac{df}{dx} = \lim_{m \rightarrow \infty} D(f, m) = \lim_{m \rightarrow \infty} D(f, m)$ whenever $\frac{df}{dx}$ is defined. For an fixed m , note that $f_n(x+1/m) \rightarrow f(x+1/m)$ in L^2 as $n \rightarrow \infty$, since by translation invariance of μ , $\int_{x \in \mathbb{R}} |f_n(x+1/m) - f(x+1/m)|^2 d\mu = \int_{x \in \mathbb{R}} |f_n(x) - f(x)|^2 d\mu \rightarrow 0$. Thus $D(f_n, m) \rightarrow D(f, m)$ in L^2 as $n \rightarrow \infty$.

We would like to show that $\frac{df}{dx} = g$ (μ -a.e.). One thing that would suffice to show this is if $\int_{\mathbb{R}} |\frac{df}{dx} - g|^2 d\mu = 0$. Unfortunately I don't know how to use the facts gathered to show this.

20/20 ✓ (c) First, suppose $f, g \in M := L^2((-1, 1)) \cap C((-1, 1))$ (i.e., L^2 w.r.t. Lebesgue measure, denoted μ) and $\alpha \in \mathbb{C}$. Then, $\|f + \alpha g\|_2 \leq \|f\|_2 + \|\alpha g\|_2$ (Minkowski) $= \|f\|_2 + |\alpha| \cdot \|g\|_2 < \infty$, so $f + \alpha g \in L^2((-1, 1))$. Also, $f + \alpha g$ is continuous by the usual closure properties of continuous functions. So $f + \alpha g \in L^2((-1, 1)) \cap C((-1, 1))$, and M is a subspace of $L^2((-1, 1))$.

M is not closed, however, as the following example shows. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be the sign function: $f(x) := x/|x|$ if $x \neq 0$, and $f(0) := 0$. f is not continuous, hence not in M . (Technically, we need to argue that the μ -a.e.-equivalence class of f , that is, the class of functions $\tilde{f}$ which equal f μ -a.e., has no representative in M . This is simple: if $\tilde{f} = f$ μ -a.e., then for every $\varepsilon > 0$ there exists an $x \in (-\varepsilon, 0)$ where $\tilde{f}(x) = f(x) = -1$; similarly, for every $\varepsilon > 0$ there exists an $x \in (0, \varepsilon)$ where $\tilde{f}(x) = f(x) = 1$. So $\tilde{f}$ is discontinuous at $x = 0$ and is not in M .) On the other hand, for $n > 0$ consider $f_n : (-1, 1) \rightarrow \mathbb{R}$ given by $f_n(x) := nx$ if $|x| \leq 1/n$, $f_n(x) := x/|x|$ if $|x| > 1/n$. Clearly $f_n \in M$, since f_n is continuous