

HOMEWORK 9.2B SOLUTION

In this quick note I will sketch a solution to HW9.2b. The problem as stated was poorly worded (and too hard) with insufficient hints, so the grader gave more or less full credit for any attempt, but I thought it would be instructive to have some brief indication of how to understand the space H of L^2 functions with L^2 derivatives (commonly written $H^1(\mathbb{R})$ and called a *Sobolev space*).

Step 1: The definition of L^2 derivative. If $g \in L^2(\mathbb{R})$, we know from class that g is equivalent to a bounded linear functional $\Lambda_g : L^2(\mathbb{R}) \rightarrow \mathbb{C}$ defined by

$$\Lambda_g(f) = \int_{\mathbb{R}} f \bar{g} dx.$$

That is, if we know the value of $\Lambda_g(f)$ for every $f \in L^2(\mathbb{R})$, we know g . Hence g is completely determined as an $L^2(\mathbb{R})$ function by the set $\{\langle f, g \rangle : f \in L^2(\mathbb{R})\}$. Moreover, if $B \subset L^2(\mathbb{R})$ is a dense subset, then g is determined by the set $\{\langle f, g \rangle : f \in B\}$.

Claim: The set $C_c^\infty(\mathbb{R})$ of C^∞ functions on $\mathbb{R}$ with compact support is dense in $L^2(\mathbb{R})$.

Proof: We showed already that the set $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, so it suffices to show that $C_c^\infty(\mathbb{R})$ is dense in $C_c(\mathbb{R})$, but this follows from the Stone-Weierstrass theorem (from 18.100).

This means that the set of values $\{\langle f, g \rangle : f \in C_c^\infty(\mathbb{R})\}$ determines the $L^2(\mathbb{R})$ function g . Now if g is differentiable and in $L^2(\mathbb{R})$, integration by parts implies for any $f \in C_c^\infty(\mathbb{R})$,

$$\int g' f dx = - \int g f' dx.$$

There are no boundary terms because g has compact support. By density of differentiable functions in $L^2(\mathbb{R})$, this formula has to hold for $g \in L^2(\mathbb{R})$. That is, we say $h \in L^2(\mathbb{R})$ is the L^2 derivative of g , $h = g'$, if

$$\int h f dx = - \int g f' dx \quad \forall f \in C_c^\infty(\mathbb{R}).$$

Furthermore, if $h \in L^2$, then by Hölder's inequality, we have

$$\left| \int h f dx \right| \leq \|h\|_{L^2} \|f\|_{L^2},$$

so that this integral is bounded by (something depending on h) times $\|f\|_{L^2}$ and not $\|f'\|_{L^2}$ is an additional requirement as well.

Step 2: Convergence in H . If $\{f_n\}$ is a Cauchy sequence in H , it is a Cauchy sequence in L^2 , so there is $f \in L^2$ such that $f_n \rightarrow f$. The sequence $\{f'_n\}$ is also Cauchy in L^2 , so there exists $g \in L^2$ such that $f'_n \rightarrow g$. We have to show that $f' \in L^2$ and that $f' = g$ in L^2 .

Claim: $f' \in L^2$.

Proof: We have to show

$$\left| \int f \phi' dx \right| \leq C \|\phi\|_{L^2}, \quad \forall \phi \in C_c^\infty(\mathbb{R}),$$

where the constant $C > 0$ necessarily depends on f . The point is that we need to estimate in terms of ϕ , not ϕ' . Let us write difference quotients for $|s| \leq 1$ and $\phi \in C_c^\infty(\mathbb{R})$:

$$\psi_s(x) = \frac{\phi(x+s) - \phi(x)}{s} = \phi'(x) + E(x, s),$$

where $E(x, s)$ is smooth, compactly supported in x , uniformly bounded, and $E(x, s) \rightarrow 0$ as $|s| \rightarrow 0$. Then

$$\left| \int f \phi' dx \right| \leq \left| \int f \psi_s dx \right| + C \max |E(x, s)| \|f\|_{L^2},$$

where C depends on the support of E in x . For $s \neq 0$, we can change variables in the first expression to get

$$\begin{aligned} \left| \int f \phi' dx \right| &\leq \left| \int \frac{f(x-s) - f(x)}{s} \phi dx \right| + C \max |E(x, s)| \|f\|_{L^2} \\ &\leq \left\| \frac{f(x-s) - f(x)}{s} \right\|_{L^2} \|\phi\|_{L^2} + C \max |E(x, s)| \|f\|_{L^2}, \end{aligned}$$

so if we can show

$$\sup_{|s| \leq 1} \left\| \frac{f(x-s) - f(x)}{s} \right\|_{L^2}$$

is bounded then taking $|s| \rightarrow 0$ the second term vanishes.

Then for $|s| \neq 0$,

$$\left\| \frac{f(x-s) - f(x)}{s} \right\|_{L^2} \leq \frac{1}{|s|} (\|f(x-s) - f_n(x-s)\|_{L^2} + \|f_n(x-s) - f_n(x)\|_{L^2} + \|f_n(x) - f(x)\|_{L^2}).$$

By translation invariance of Lebesgue measure, the first and third terms are equal, and by taking n sufficiently large, we can assume

$$(1) \quad \|f_n - f\|_{L^2} \leq |s|/2.$$

A similar argument using difference quotients implies that, since $f_n \in H$,

$$\left\| \frac{f_n(x-s) - f_n(x)}{s} \right\|_{L^2} \leq C_n,$$

independent of $|s| \leq 1$, and the constants C_n are bounded in n because the sequence $\{f_n\}$ is Cauchy in H . Hence if n is chosen sufficiently large that (1) holds, we have

$$\begin{aligned} &\left\| \frac{f(x-s) - f(x)}{s} \right\|_{L^2} \\ &\leq \frac{1}{|s|} (\|f(x-s) - f_n(x-s)\|_{L^2} + \|f_n(x-s) - f_n(x)\|_{L^2} + \|f_n(x) - f(x)\|_{L^2}) \\ &\leq C, \end{aligned}$$

with $C > 0$ independent of n and s . Hence $f \in H$.

Claim: $f' = g$ as L^2 functions.

Proof: This is easy with our “weak derivative” idea, and continuity of the L^2 inner product. We have $f' = g$ in L^2 if and only if

$$\langle f', \phi \rangle_{L^2} = \langle g, \phi \rangle_{L^2}$$

for every $\phi \in C_c^\infty(\mathbb{R})$. Fix such a ϕ and apply continuity in the first component of the inner product:

$$\begin{aligned}\langle f', \phi \rangle &= -\langle f, \phi' \rangle \\ &= -\lim_{n \rightarrow \infty} \langle f_n, \phi' \rangle \\ &= \lim_{n \rightarrow \infty} \langle f'_n, \phi \rangle \\ &= \langle g, \phi \rangle.\end{aligned}$$