

H , we have

$$\begin{aligned}\|w\|^2 &= (w, w) = \sum_{i \leq m} \alpha_i \overline{\alpha_j} (v_i, v_j) \\ &= \sum_{i \leq m} |\alpha_i|^2 \quad (\text{by orthonormality}) \\ &= \sum_{i \leq m} |(w, v_i)|^2 \\ &= \|\phi(w)\|^2.\end{aligned}$$

This shows that $\phi(w) = 0$ iff $w = 0$, so ϕ is injective; and it shows that ϕ has operator norm 1. Also, we verify that ϕ is invertible, with the formula $\phi^{-1}(\alpha_1, \dots, \alpha_j) = \sum_{i \leq m} \alpha_i v_i \in M$ (so that in particular ϕ has image equal to all of $\mathbb{C}^N$), and

$$\begin{aligned}\|\phi^{-1}(\alpha_1, \dots, \alpha_j)\|^2 &= \left(\sum_{i \leq m} \alpha_i v_i, \sum_{i \leq m} \alpha_i v_i \right) \\ &= \sum_{i \leq m} |\alpha_i|^2 \\ &= \|(\alpha_1, \dots, \alpha_j)\|^2.\end{aligned}$$

Now suppose $\{w_n\}_{n>0}$ is a Cauchy sequence in M . Since ϕ is a bounded linear map, $\{\phi(w_n)\}$ is Cauchy in $\phi(M) = \mathbb{C}^m$. By completeness of $\mathbb{C}^m$, $\{\phi(w_n)\} \rightarrow v$ for some $v \in \mathbb{C}^m$. By the boundedness of ϕ^{-1} , we can apply ϕ^{-1} to both sides to get $\{w_n\} = \{\phi^{-1}(\phi(w_n))\} \rightarrow \phi^{-1}(v) \in M$. Thus $\{w_n\}$ converges in M , and M is closed.

(b) First we show H is a subspace of $L^2(\mathbb{R})$. Clearly if $f, g \in H \subset L^2(\mathbb{R})$ then $f + \alpha g \in L^2(\mathbb{R})$. Also, we have

$$\begin{aligned}\left\| \frac{d}{dx}(f + \alpha g) \right\|_2 &= \left\| \frac{df}{dx} + \alpha \frac{dg}{dx} \right\|_2 \\ &\quad (\text{linearity of } d/dx) \\ &\leq \left\| \frac{df}{dx} \right\|_2 + |\alpha| \cdot \left\| \frac{dg}{dx} \right\|_2 \\ &\quad (\text{Minkowski}) \\ &< \infty,\end{aligned}$$

since $f, g \in H$.

The function $f(x) = e^{-|x|}$ is not differentiable at zero, since $\lim_{x \rightarrow 0^+} (f(x) - f(0))/x = \lim_{x \rightarrow 0^+} (e^{-x} - e^0)/x = \frac{d}{dx}(e^{-x}) = -e^0 = -1$, while $\lim_{x \rightarrow 0^-} (f(x) - f(0))/x = \lim_{x \rightarrow 0^-} (e^{-x} - e^0)/x = \frac{d}{dx}(e^x) = e^0 = 1$.

We have $f \equiv e^{-x}$ on $(0, \infty)$, so $\frac{df}{dx} \equiv -e^{-x}$ on $(0, \infty)$. Similarly $f \equiv e^x$ on $(-\infty, 0)$, so $\frac{df}{dx} \equiv e^x$ on $(-\infty, 0)$, and $\|f\|_{L^2(\mathbb{R})}^2 = \int_{(-\infty, 0)} e^{2x} d\mu + \int_{(0, \infty)} e^{-2x} d\mu = \frac{1}{2}[e^{2x}]_{-\infty}^0 - \frac{1}{2}[e^{-2x}]_0^\infty =$