

18.125, Homework 9

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20/20 ✓ **9.1** First, let $M \leq H$ be any subspace of a Hilbert space H . (We use $U \leq V$ to denote that U is a subspace of V .) Suppose $v \in M$ and take any $w \in M^\perp$. Then by definition of $M^\perp$, we have $(w, v) = 0$. Thus, $v \in (M^\perp)^\perp$, and so

$$M \leq (M^\perp)^\perp.$$

Equality need not hold in general. For instance, let $H := \ell^2(\mathbb{N})$, the Hilbert space of square-summable sequences. Let $M := c_0$, the subspace of all sequences with finitely many nonzero terms. First, note that $M^\perp = \{0\}$. For, if $v = (v_1, v_2, \dots) \in M^\perp$, and we let $e_i \in c_0$ be the sequence which equals 1 in the i -th term and 0 elsewhere, then $0 = (v, e_i) = v_i$, so that v is the 0 element in H .

Then, $(M^\perp)^\perp = \{0\}^\perp = H \neq M$.

But now suppose M is a *closed* subspace of H , and take any $x \in (M^\perp)^\perp$. Since M is closed, by our 'orthogonal projection' result we can write $x = y + z$, for some pair $y \in M, z \in M^\perp$. Then we compute $(x, z) = (y + z, z) = (y, z) + (z, z) = 0 + (z, z)$ (since $z \in M^\perp$) $= (z, z)$. But also, $(x, z) = 0$, since $x \in (M^\perp)^\perp$. Thus $(z, z) = 0$, so that $z = 0$ and $x = y \in M$. So in this case we have the inclusion $M \leq (M^\perp)^\perp$, which with our earlier work implies $M = (M^\perp)^\perp$.

20/20 ✓ **9.2** A Hilbert space H is by definition complete, so to show that a subspace $M \leq H$ is closed it is always enough to show that Cauchy sequences in M converge to limit points in M .

(a) Let M be any subspace of $\mathbb{C}^N$. M is trivially closed if $M = \{0\}$ or $M = \mathbb{C}^N$ (since $\mathbb{C}^N$ is a Hilbert space), so assume otherwise. Then there exists $1 \leq m < N$ and orthonormal vectors $u_1, \dots, u_m$ which span M , along with an orthonormal basis $v_1, \dots, v_{N-m}$ for $M^\perp$.

Define a linear map $\phi : \mathbb{C}^N \rightarrow \mathbb{C}^m$ by $\phi(w) := ((w, v_1), \dots, (w, v_m))$. If $w = \sum_{i=1}^N \alpha_i v_i \in$