

2 Problem 2

(a) Using Holder's inequality and the fact that $\mu(X) = 1$,

$$\|f\|_{L^r} = \left(\int |f|^r d\mu \right)^{1/r} \leq \left(\left(\int (|f|^r)^{s/r} d\mu \right)^{r/s} \left(\int d\mu \right)^{1-r/s} \right)^{1/r} = \left(\int |f|^s d\mu \right)^{1/s} = \|f\|_{L^s}, \text{ as required. } \blacksquare$$

(b) Part (a) immediately establishes the claim since everything which has a finite L^s norm also has a finite L^r norm bounded by its L^s norm. Now, we claim that, given a measure space X with finite total measure $\mu(X) < \infty$, every L^p space is equal if and only if there exists some "minimum measure" - i.e. there exists some number k such that for every measurable set E , $\mu(E) \neq 0$ implies $\mu(E) > k$. First, suppose that the space has this property. Then, given arbitrary f , there are two possibilities: either it is bounded, or it is not. If it's bounded, then by finiteness of the measure space, $\int |f|^p d\mu \leq \sup |f|^p \mu(X) < \infty$. Thus, f is in every L^p space. If it is not bounded, then $\mu(\{f = \infty\}) \neq 0$, so $\mu(\{f = \infty\}) > k$, so f does not belong to any L^p space because $\int |f|^p d\mu > \infty$ for every $p > 0$. Conversely, suppose that the space does not have the "minimum measure" property. Then, choose disjoint sets E_i such that $\mu(E_i) < 1/2^i$. Then, given $0 < r < s < \infty$, the function $f = \sum_{i=1}^{\infty} \mu(E_i)^{-1/s} \chi_{E_i}$ has L^p norm given by $\sum \mu(E_i)^{1-p/s}$ and therefore will belong to L^r but not L^s . If $s = \infty$, then taking $f = \sum_{i=1}^{\infty} \mu(E_i)^{-1/2r} \chi_{E_i}$ will belong to L^r and not L^∞ . This establishes the converse. $\blacksquare$

(c) We treat the $\log f \in L_1$ case first. Since the exponential function is convex, we may apply Jensen's inequality:

$$\exp \left(\int \log |f| d\mu \right) = \left(\exp \left(\int \log (|f|^p) d\mu \right) \right)^{1/p} \leq \left(\int |f|^p d\mu \right)^{1/p} = \|f\|_{L^p}$$

Letting p tend to 0, this establishes one direction. For the other direction, we use the fact that $\log(y) \leq y - 1$ for all positive y (this follows directly from Taylor expansion for sufficiently small y and monotonicity):

$$\log \|f\|_{L^p} = \log \left(\int |f|^p d\mu \right)^{1/p} \leq \frac{1}{p} \left(\int |f|^p d\mu - 1 \right)$$

But, $\mu(X) = 1$, so we can move the factor of 1 into the integral and obtain

$$\log \|f\|_{L^p} \leq \frac{1}{p} \int (|f|^p - 1) d\mu$$

Now, the function $g(p) = (|f|^p - 1)/p$ has derivative $(p|f|^p \log |f| - |f|^p + 1)/p^2 > 0$ near $p = 0$, so $g(p)$ decreases as p tends to 0. Thus, given any r sufficiently small, $g(p)$ is dominated by $(|f|^r - 1)/r$ for all $p < r$. Hence, we may take the limit as $p \rightarrow 0$ of both sides and apply Lebesgue's dominated convergence theorem and L'Hospital's rule:

$$\lim_{p \rightarrow 0} \log \|f\|_{L^p} \leq \lim_{p \rightarrow 0} \int \frac{|f|^p - 1}{p} d\mu = \int \lim_{p \rightarrow 0} \frac{|f|^p - 1}{p} d\mu = \int \log |f| d\mu$$

from which the required inequality is obtained by exponentiation (this is permissible since $\exp$ is continuous everywhere).

Now, we deal with the case that $\log f \notin L_1$ so that $\int \log |f| d\mu = -\infty$ - i.e. the support of f has measure less than $\mu(X) = 1$. Then, we need to show that $\lim_{p \rightarrow 0} \|f\|_{L^p} = 0$. To this end, we approximate f by simple functions $f_n \nearrow f$. Write $f_n = \sum c_i \chi_{E_i}$, where the E_i are disjoint sets. Then, $\lim_{p \rightarrow 0} \|f_n\|_{L^p} = \lim_{p \rightarrow 0} \sum_i (c_i^p \mu(E_i))^{1/p} \leq \lim_{p \rightarrow 0} \max(c_i) \mu(\bigcup E_i)^{1/p}$. But $\mu(\bigcup E_i)^{1/p} < \mu(X) < 1$, so this limit is just 0, as required. By non-negativity, however, we have $\|f_n\| \geq 0$, so $\|f_n\| = 0$. Now, by the monotone convergence theorem we may pass to the limit as $n \rightarrow \infty$ so that $\lim_{p \rightarrow 0} \|f\|_{L^p} = 0$, as required. $\blacksquare$

3 Problem 3

Consider the measure space $X = \mathbb{Z}$ with μ the counting measure, and consider functions from $(X, \mu) \rightarrow \mathbb{R}$. These are countable sequences of real numbers $\{a_i\}$ and the L_p norm of such a sequence is just $\sum a_i^p$. If $r < s$ and the