

Let $p = \lambda r + (1-\lambda)s$, $\lambda \in (0,1)$.
 d) We have shown in b) that

$$\log(\phi(p)) \leq \lambda \log(\phi(r)) + (1-\lambda) \log(\phi(s))$$

$$\begin{aligned} \text{Thus } \log(\|f\|_p) &= \frac{1}{p} \log(\phi(p)) \leq \frac{1}{p} (\lambda r \log(\|f\|_r) + (1-\lambda)s \log(\|f\|_s)) \\ &\leq \frac{1}{p} (\lambda r + (1-\lambda)s) \max\{\log(\|f\|_r), \log(\|f\|_s)\} \\ &= \max\{\log(\|f\|_r), \log(\|f\|_s)\} \end{aligned}$$

$$\text{Hence } \|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}.$$

$$\forall f \in L^r \cap L^s, \|f\|_r < \infty, \|f\|_s < \infty \Rightarrow$$

$$\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\} < \infty \Rightarrow \|f\|_p < \infty \Rightarrow f \in L^p$$

e) Case 1. $\|f\|_\infty = +\infty$.

Then $\mu(A_n := \{x \in X : |f(x)| > n\}) > 0 \Rightarrow \lim_{n \rightarrow \infty} \mu(A_n) > 0$ exists or $\mu(A_n) \rightarrow \infty$

$$\forall p > 0, \quad n \mu(A_n)^{1/p} \leq \left(\int_X |f|^p d\mu \right)^{1/p} = \|f\|_p$$

$$\text{Let } n \rightarrow \infty \Rightarrow \|f\|_p \rightarrow \infty$$

Case 2. $0 < \|f\|_\infty < \infty$ $\|f\|_p > 0, \forall p > 0$

Let $F = \frac{f}{\|f\|_\infty}$, then $\|F\|_\infty = 1$. $\Rightarrow \mu(\{x \in X : F(x) > 1\}) = 0$.

$$0 < \|F\|_p^p = \int_X |F|^p d\mu \leq \int_X |F|^r d\mu = \|F\|_r^r = \frac{\|f\|_r^r}{\|f\|_\infty^r} < \infty$$

$$\Rightarrow \|F\|_p \leq \|F\|_r^{r/p} \rightarrow 1 \text{ as } p \rightarrow \infty \Rightarrow \lim_{p \rightarrow \infty} \|F\|_p \leq 1.$$

Since $\|F\|_\infty = 1$, $\forall \varepsilon \in (0,1), m_\varepsilon = \mu(\{x \in X : |f(x)| > 1-\varepsilon\}) > 0$, therefore

$$0 < (1-\varepsilon) m_\varepsilon^{1/p} \leq \|F\|_p$$

$$\text{Let } p \rightarrow \infty \Rightarrow \|F\|_p \geq 1-\varepsilon$$

$$\text{Let } \varepsilon \rightarrow 0 \Rightarrow \|F\|_p \geq 1 \text{ as } p \rightarrow \infty$$

$$\text{Hence } \|F\|_p \rightarrow 1 \text{ as } p \rightarrow \infty.$$

$$\|f\|_p = \|F\|_p \|f\|_\infty \rightarrow \|f\|_\infty \text{ as } p \rightarrow \infty$$