

Then $\int_0^\epsilon g_a(x) dx \geq \int_0^\epsilon \frac{1}{x} dx = \infty \Rightarrow \int_0^\epsilon g_a(x) dx = \infty$

$\Rightarrow g_a \notin L^1$

Hence $g_a \in L^1 \Leftrightarrow a \geq 1$. Thus $g_a \in L^p \Leftrightarrow p \leq a$. (3)

$\int_2^\infty G_1(x) dx = \int_2^\infty \left(-\frac{1}{\log x}\right) dx = \frac{1}{\log 2} \Rightarrow G_1 \in L^1$

For $a < 1$, $x \log^2 x \begin{cases} > 2 \log^2 2 \\ > e \log^2 e = e > 1, x > e \end{cases}, 2 < x \leq e \Rightarrow G_a(x) = (x \log^2 x)^{-\frac{1}{a}} \begin{cases} < (2 \log^2 2)^{-\frac{1}{a}}, 2 < x \leq e \\ < (x \log^2 x)^{-1}, x > e \end{cases}$
 $= G_1(x)$

Thus

$\int_2^e G_a(x) dx < \infty$ and $\int_e^\infty G_a(x) dx \leq \int_e^\infty G_1(x) dx < \infty$.

Hence $G_a \in L^1$.

For $a > 1$, since $\lim_{x \rightarrow \infty} \frac{x \log^2 x}{x^a} = \lim_{x \rightarrow \infty} \frac{\log^2 x}{x^{a-1}} = 0, \exists m > 2$ s.t.
 $\forall x > m, x^a > x \log^2 x$. Then $x^{-1} < (x \log^2 x)^{-\frac{1}{a}} = G_a(x)$.

$\int_m^\infty G_a(x) dx \geq \int_m^\infty x^{-1} dx = \infty \Rightarrow G_a \notin L^1$

Hence $G_a \in L^1 \Leftrightarrow a \leq 1$. Therefore $G_a \in L^p \Leftrightarrow p \geq a$. (4)

With the above results (1) & (4), we construct f for any connected subset E of $(0, \infty)$:

$(a, b): f = f_b + f_a; (a, \infty): f_a; (0, b): f_b;$

$[a, b]: f = g_b + G_a; [a, \infty): G_a; (0, b]: g_b;$

$[a, b): f = f_b + G_a; (a, b]: g_b + f_a;$

$\{a\}: f = g_a + G_a; \emptyset: f_a + G_a$

~~This also works, but your technique applies here too.~~